A two-dimensional disturbed flows over a flat plate: theoretical and numerical approach

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Abstract. The incompressible 2D flow over a flat plate with and without incidence is studied in respect of the propagation of spatial mode disturbances, by solving the unsteady Navier-Stokes equations. The solutions are compared with a theoretical analysis providing an analytical uniformly valid approach. The effect of the flow disturbances is studied by analytical and the numerical approaches.

Key words: 2D disturbances, flat plate, incompressible flow.

1. Introduction

Following the ideas of Libby and Fox [7], a theoretical analysis of algebraic disturbances evolving spatially in the Blasius flow is lead. These disturbances, searched as self similar two-dimensional disturbances in the boundary layer, are built in the whole flow domain with the asymptotic method of the matched expansions for high Reynolds numbers. They are sought as a product of functions of power of x and functions of the similarity variable. For particular case without incidence of the Blasius flow, the first two-dimensional mode, named *Stewartson* mode, is retrieved. This asymptotic method, previously used in order to obtain a uniformly valid approximation for the mean Falkner-Skan flow [9], allows taking into account to the non-parallel effects of the disturbed flow and ensures the irrotationnality of the flow outside the boundary layer. The good agreement between numerical and theoretical solutions has been proved earlier [10]. An extension of this method to the disturbed flow is presented in this paper ; it allows putting in evidence generic disturbances of this flow.

2. 2D disturbances over a flat plate

The uniformly valid disturbed solution over a flat plate

In this section, we consider a viscous bidimensional incompressible flow perturbations over a flat plate with an incidence angle φ. We are interested particularly in the uniformly valid perturbed solution provided in [11]. By introducing the parameter $\varepsilon = \frac{1}{\sqrt{re}}$, one can write the semi-analytical solution for the wall-normal and streamwise velocity perturbation components ($u_{p_{th}}$,$v_{p_{th}}$) and the pressure perturbation ($p_{p_{th}}$) in the form:

$$\begin{aligned} u_{p_{th}} &= x^{\lambda}h'_n(\eta)+\varepsilon A_{\lambda}(\frac{n-1}{2}-\lambda)\rho^{\lambda-\frac{n+1}{2}}\sin((\lambda-\frac{n+1}{2})\theta) \\ v_{p_{th}} &= \varepsilon x^{\lambda-\frac{n+1}{2}}\left[\frac{1-n}{2}\eta h'_n(\eta)+(\frac{n-1}{2}-\lambda)(h_n(\eta)-A_{\lambda})\right] \\ &\quad +\varepsilon A_{\lambda}(\frac{n-1}{2}-\lambda)\rho^{\lambda-\frac{n+1}{2}}\cos((\lambda-\frac{n+1}{2})\theta) \\ p_{p_{th}} &= \varepsilon A_{\lambda}(\frac{n-1}{2}-\lambda)\rho^{\lambda-\frac{1-n}{2}}\sin((\frac{3n+1}{2}-\lambda)\theta) \end{aligned}$$

where $n=\frac{\varphi}{\pi-\varphi}$, $\theta=\arctan(y/x)$, $\rho=\sqrt{x^2+y^2}$, $\eta=\frac{1}{\varepsilon}x^{\frac{n-1}{2}}y$ and $h_n(\eta)$ is a similarity function which satisfies to the eigenvalue problem:

$$h'''_n(\eta)+\frac{(n+1)}{2}F_n(\eta)h''_n(\eta)-(n+\lambda)F'_n(\eta)h'_n(\eta)+(\lambda-\frac{n-1}{2})F''_n(\eta)h_n(\eta)=0$$

with boundary conditions:

$$h_n(0)=0 \;\; ; \;\; h'_n(0)=0 \;\; ; \;\; \lim_{\eta\to\infty}h'_n(\eta)=0 \;\; ; \;\; \lim_{\eta\to\infty}h_n(\eta)=A_{\lambda} \;\; ; \;\; h''_n(0)=1$$

λ is the eigenvalue associated to the eigenfunction $h_n(\eta)$, A_{λ} is a constant who depends on the numerical solution of the problem and $F_n(\eta)$ is the Falkner-Skan function solution of the problem:

$$F'''_n(\eta)+\frac{(n+1)}{2}F_n(\eta)F''_n(\eta)+n(1-F^2_n(\eta))=0$$

with boundary conditions:

$$F_n(0)=0 \;\; ; \;\; F'_n(0)=0 \;\; ; \;\; \lim_{\eta\to\infty}F'_n(\eta)=1$$

In Figure 1, we show the theoretical streamwise and wall-normal velocity perturbation profiles for the first mode versus the incidence angle φ at x=6 with a Reynolds number equal to 1000. These more realistic solutions are used when analysing the destabilisation of the flow, especially under the adverse pressure gradient effects of high incidence.

A numerical method to solve the Navier-Stokes equations of the disturbed flow in the vicinity of the basic flow has been developed. The vorticity-stream function formulation is used with a fourth-order compact scheme. The spatial discretisation uses the finite differences scheme with an equal mesh size in each direction and the temporal discretisation uses the Crank-Nicolson scheme which ensures an unconditional stability of the method. The resolution of the discretized equations is done by the fractional temporal step formulation adopted in [1]-[3] with an alternating implicit direction scheme. The method is second-order accurate in time and fourth-order accurate in space. Dirichlet and Neumann boundary conditions have been successfully tested, as well as a non reflecting outlet boundary condition [5]. Numerical tests have been performed on different problems such that the analytical flow solution of a Green-Taylor vortex, the lid-driven cavity flow, the Falkner-Skan flow and compared to the analytical and benchmark solutions found in the literature [1]-[4]-[6]-[9]. This method is also adapted for the linearized Navier-Stokes equations around the two-dimensional basic solution. The uniformly valid solution provided in [9] is used for the basic flow. The equations are solved in the Cartesian coordinates (x,y) in a semi-infinite domain [1,16]×[0,10] over the plate. The numerical simulation allows capturing with a very good agreement the transient stages as well as the steady-state reached solutions of the linearized (Figure 2)and non linear Navier-Stokes equations. Other comparisons will be presented at the conference.

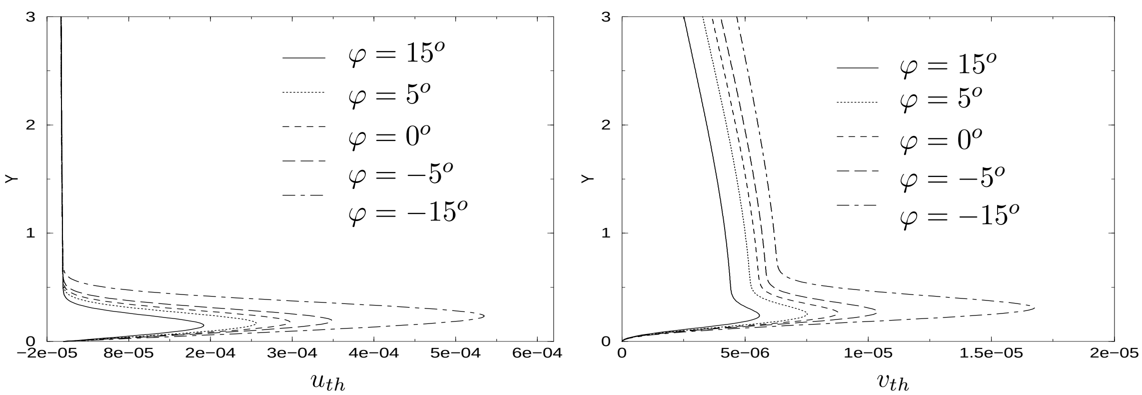

Figure 1. Disturbance streamwise and wall-normal theoretical velocity profiles for the first mode with various incidence angles φ at x=6. Re=1000 *and grid mesh of* 301 × 501

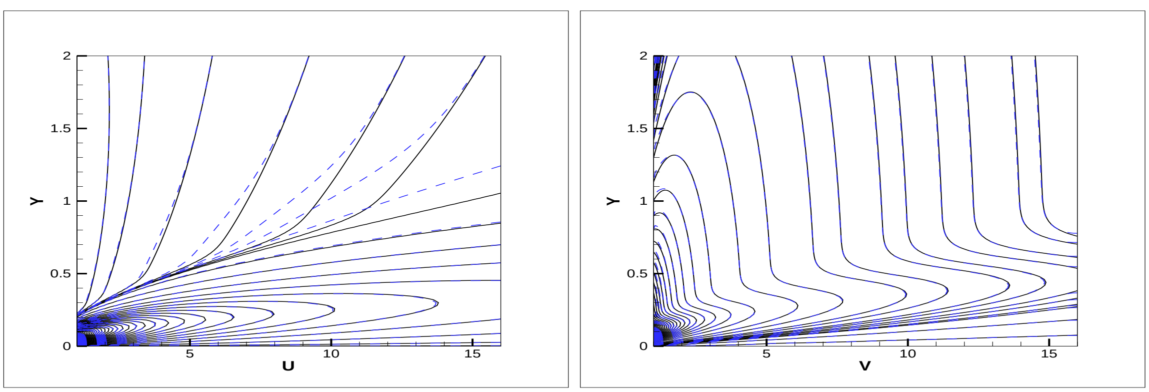

Figure 2. Disturbance streamwise and wall-normal velocity contours of the theoretical solution (solid) and the numerical solution (−−) that reached steady-state for an incidence angle $\varphi = -5$. Re=1000 *and grid mesh of* 301 × 501

2D optimal disturbances

A well-known, the transition from laminar to turbulent flow is a critical process in many engineering applications. We study the growth of bidimensional optimal disturbances over a flat plate. The aim is to optimize the initial disturbance (u_{opt},v_{opt}) at $x_{in} = 1$, the beginning of the interval, in order to achieve maximum possible amplification of the disturbance energy at $x_{out} = 16$, the end of the interval. We define the growth G over the interval $x_{in} \leq x \leq x_{out}$ as the ratio between the disturbance energy E at the end and beginning of the interval:

$$G(x_{in}, x_{out}, Re) = \frac{E(x_{out})}{E(x_{in})} \ , \ E(x) = E_u(x) + E_v(x)$$

$$E_u(x) = \frac{1}{2}\int_0^{\infty} u^2(x,y)dy \ , \ E_v(x) = \frac{1}{2}\int_0^{\infty} v^2(x,y)dy$$

The calculations of the optimal disturbances are carried out with an adjoint-based optimization procedure as in [2]-[8]. Figure 3 shows the wall-normal velocity profile and the kinetic energy growth of the optimal disturbance associated with initial zero streamwise velocity component. The maximum growth obtained $G_{max} = 1.74.10^{-6}Re$ at $x = 1.42$. The optimal disturbance is searched in a general way with both streamwise and wall-normal velocity components, as shown in Figure 4. Preliminary results give the maximum growth $G_{max} = 1.22.10^{-3}Re$ at $x = 1.37$, which is in the same order of the maximum growth found in a 3D optimal disturbances study ($G_{max} = 2.2.10^{-3}Re$, [2]) and the streamwise velocity profil of 2D optimal disturbance (Figure 4(a)) is similar to the spanwise velocity profil of the 3D optimal disturbance found in [8].

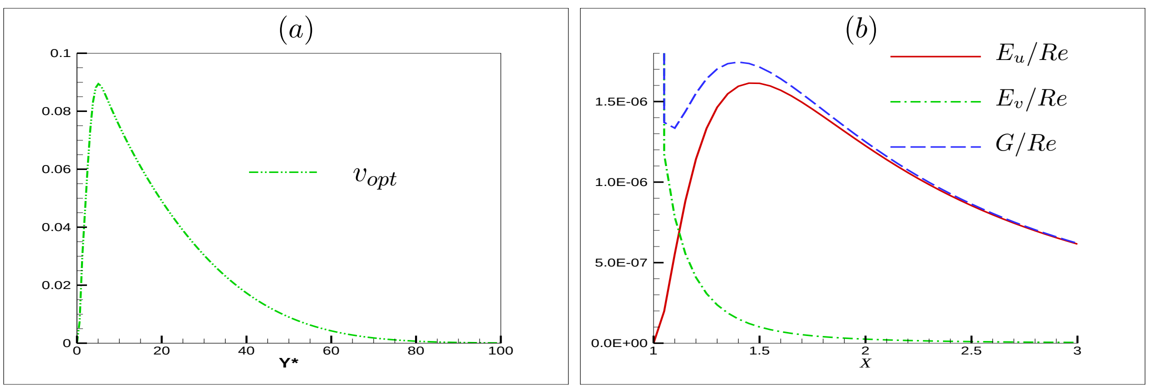

*Figure 3. Wall-normal velocity profile (a) and kinetic energy growth (b) of the 2D optimal disturbance in the case of initial zero streamwise velocity component. Re=*1000 *and grid mesh of* 301 × 501

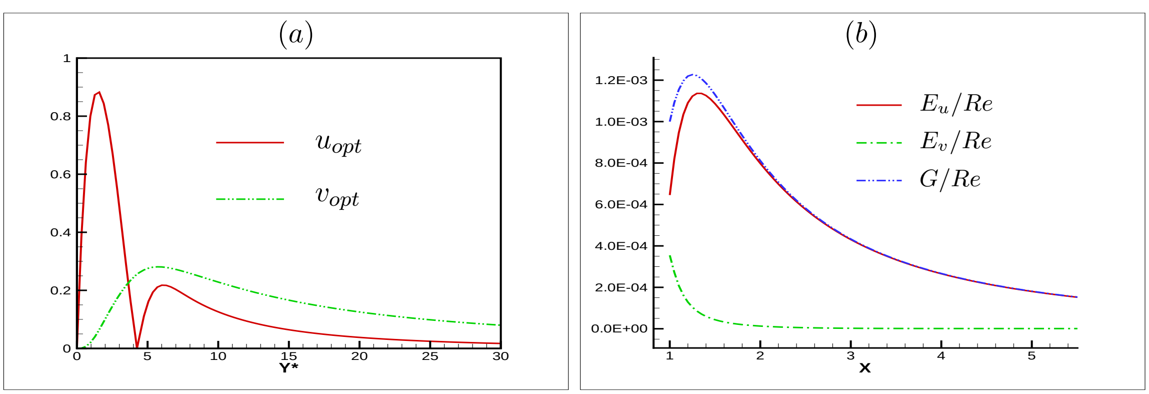

*Figure 4. Streamwise and wall-normal velocity profiles (a) and kinetic energy growth (b) of the 2D optimal disturbance. Re=*1000 *and grid mesh of* 301 × 501

3. Conclusions

A theoretical and numerical method is provided, allowing treating the problem of a flow over a semi-infinite flat plate with and without incidence by using the Navier-Stokes equations. The Navier-Stokes solver provided the time-marching solutions and is able to compute unsteady separated flows. The main objective of this study is to quantify the amplification of instabilities in this flow and their control. This numerical method is adapted to solve the adjoint Navier-Stokes equations in order to study the 2D optimal disturbances for this type of flows. Some preliminary results are compared to existing benchmark solutions found in 3D optimal disturbances studies.

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