Trefftz-Type Sensitivity Analysis for Boundary Value Problem of Poisson Equation

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Summary

This paper describes the application of the Trefftz-type boundary element method to the sensitivity analysis of the boundary value problem of Poisson equation. The unknown function is approximated with the T-complete functions of the Laplace equation and the particular solutions related to the inhomogeneous term of the Poisson equation. Direct differentiation of the function leads to the sensitivities. Finally, the present method is applied to a simple numerical example.

Introduction

This paper describes the application of the Trefftz-type boundary element method to the sensitivity analysis of the boundary value problem of Poisson equation.

Firstly, the Trefftz formulation for the boundary value problem of Poisson equation is formulated. In the formulation, the inhomogeneous term of Poisson equation is approximated with the polynomial function in Cartesian coordinates to derive the related particular solution. The use of the particular solution transforms the boundary value problem of the Poisson equation into that of the Laplace equation. Since the unknown parameters included into the particular solution depends on the unknown function, the derived boundary value problem is solved by the iterative process.

The unknown function is approximated with the T-complete function and the particular solution. Direct differentiation of the function leads to the sensitivities. The boundary-specified value and the shape parameter are taken as the variable for the sensitivity analysis and then, the sensitivity analysis formulations are described.

Trefftz Formulation for Poisson Equation

We shall consider the governing equation and the boundary conditions of the boundary value problem of Poisson equation given as

$$\nabla^2 u + b(x,y,u) = 0 \quad (\text{in } \Omega) \tag{1}$$

$$u = \bar{u} \quad (\text{on } \Gamma_u) \ , \ q = \bar{q} \quad (\text{on } \Gamma_q) \tag{2}$$

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where Ω, Γ_u and Γ_q denote the object domain under consideration and its potential u- and flux q-specified boundaries, respectively.

We shall approximate the inhomogeneous term b with a polynomial as follows.

$$b = c_1 + c_2 x + \cdots + c_{21} y^5 \equiv c_1 r_1 + c_2 r_2 + \cdots + c_{21} r_{21} = \boldsymbol{c}^T \boldsymbol{r} \tag{3}$$

In this study, the term b is approximated with the 5-order polynomial. The use of the equation (3) transforms the original governing equation as follows.

$$\nabla^2 u + \boldsymbol{c}^T \boldsymbol{r} = 0 \tag{4}$$

Since the term r_i is the term of the polynomial function, the related particular solution u_i^p can be determined easily.

In the Trefftz method, the homogeneous solution of the governing equation u^h is approximated with the superposition of the related T-complete function u_i^*[1]. The unknown function u is approximated with the T-complete function u_i^* and the particular solution u_i^p as follows.

$$u = u^h + \boldsymbol{c}^T \boldsymbol{u}^p = \boldsymbol{a}^T \boldsymbol{u}^* + \boldsymbol{c}^T \boldsymbol{u}^p \tag{5}$$

where $\boldsymbol{a}$ denotes the unknown parameter vector for approximating the homogeneous solution.

Equation (5) satisfies Eq.(1) but dose not satisfy Eq.(2). Substituting Eq.(5) to (2) leads to the residual expressions. The residual equations are satisfied at the boundary collocation points by means of the collocation method. We have

$$\boldsymbol{K}\boldsymbol{a} = \boldsymbol{f} - \boldsymbol{B}\boldsymbol{c} \tag{6}$$

The unknown parameter vector $\boldsymbol{c}$ in Eq.(6) is determined by the iterative process. Equation (3) held at the iteration step (k) and $(k+1)$ is as follows.

$$b^{(k+1)} = \boldsymbol{r}^T \boldsymbol{c}^{(k+1)} \quad , \quad b^{(k)} = \boldsymbol{r}^T \boldsymbol{c}^{(k)}$$

Subtracting both sides of the above equations leads to

$$b^{(k+1)} - b^{(k)} = \boldsymbol{r}^T (\boldsymbol{c}^{(k+1)} - \boldsymbol{c}^{(k)}) \equiv \boldsymbol{r}^T \Delta \boldsymbol{c} \tag{7}$$

where the superscript (k) denotes the number of iteration.

The collocation points referred as "the computing point" are placed on the boundary and within the domain. Holding Equation (7) at the computing point and arranging them in the matrix form, we have

$$\boldsymbol{D}\Delta\boldsymbol{c} = \boldsymbol{f} \tag{8}$$

where $\boldsymbol{D}$ and $\boldsymbol{f}$ denote the coefficient matrix and vector, respectively. Once equation (8) is solved for $\Delta\boldsymbol{c}$ with the singular value decomposition, the parameter $\boldsymbol{c}$ is updated with

$$\boldsymbol{c}^{(k+1)} = \boldsymbol{c}^{(k)} + \Delta\boldsymbol{c} \tag{9}$$

The convergence criterion is defined as

$$\eta \equiv \frac{1}{M_c}\sum_{i=1}^{M} |\Delta b(Q_i)| < \eta_c \tag{10}$$

where M_c and η_c denote the total number of the computing points and the positive constant specified by an user, respectively.

Sensitivity Analysis for Specified Value on Boundary

We shall consider here that the inhomogeneous term depends only on the unknown function u; i.e., $b = b(x, y, u)$.

Direct differentiation of Eq.(5) with respect to the specified value leads to

$$\dot{u} = \dot{\boldsymbol{a}}^T\boldsymbol{u}^* + \dot{\boldsymbol{c}}^T\boldsymbol{u}^p \tag{11}$$

where $(\dot{\ })$ denotes the differentiation with respect to the specified value. Holding Eq.(11) at all computing points, we have

$$\boldsymbol{K}_1\dot{\boldsymbol{a}} + \boldsymbol{B}_1\dot{\boldsymbol{c}} - \boldsymbol{I}\dot{\boldsymbol{u}} = 0 \tag{12}$$

Since the matrix $\boldsymbol{K}$ and the vector $\boldsymbol{B}$ do not depend on the specified value, direct differentiation of Eq.(6) with respect to the variable leads to

$$\boldsymbol{K}\dot{\boldsymbol{a}} + \boldsymbol{B}\dot{\boldsymbol{c}} = \dot{\boldsymbol{f}} \tag{13}$$

Direct differentiation of Eq.(3) with respect to the specified value leads to

$$\dot{\boldsymbol{c}}^T\boldsymbol{r} - \frac{\partial b}{\partial u}\dot{u} = 0$$

Holding this equation at all computing points, we have

$$\boldsymbol{G}\dot{\boldsymbol{c}} - \boldsymbol{H}_1\dot{\boldsymbol{u}} = 0 \tag{14}$$

Equations (12), (13) and (14) are collected in the system of equations, which are solved for $\dot{\boldsymbol{u}}$.

Sensitivity Analysis for Shape Parameter

Direct differentiation of Eq.(5) with respect to the parameter, we have

$$\dot{\boldsymbol{a}}^T\boldsymbol{u}^* + \dot{\boldsymbol{c}}^T\boldsymbol{u}^p - \dot{u} = -(\boldsymbol{a}^T\dot{\boldsymbol{u}}^* + \boldsymbol{c}^T\dot{\boldsymbol{u}}^p) \tag{15}$$

where $(\dot{\ })$ denotes the differentiation with respect to the parameter. Holding Eq.(15) at the computing points, we have

$$\boldsymbol{K}_1\dot{\boldsymbol{a}} + \boldsymbol{B}_1\dot{\boldsymbol{c}} - \boldsymbol{I}\dot{\boldsymbol{u}} = \boldsymbol{g}_1 \tag{16}$$

Direct differentiation of Eq.(6) with respect to the parameter, we have

$$\boldsymbol{K}\dot{\boldsymbol{a}} + \boldsymbol{B}\dot{\boldsymbol{c}} = \dot{\boldsymbol{f}} - \dot{\boldsymbol{K}}\boldsymbol{a} - \dot{\boldsymbol{B}}\boldsymbol{c} \equiv \boldsymbol{g} \tag{17}$$

Direct differentiation of Eq.(3) with respect to the parameter, we have

$$\dot{\boldsymbol{c}}^T\boldsymbol{r} - \frac{\partial b}{\partial u}\dot{u} = -\boldsymbol{c}^T\dot{\boldsymbol{r}}$$

Holding this equation at the computing points, we have

$$\boldsymbol{G}\dot{\boldsymbol{c}} - \boldsymbol{H}_1\dot{\boldsymbol{u}} = \boldsymbol{g}_2 \tag{18}$$

Equations (16), (17) and (18) are collected in the system of equations, which are solved for $\dot{\boldsymbol{u}}$.

Example and Discussion

An object domain and boundary conditions are shown in Fig.1. A governing equation is given as follows.

$$\nabla^2 u + u = 0 \tag{19}$$

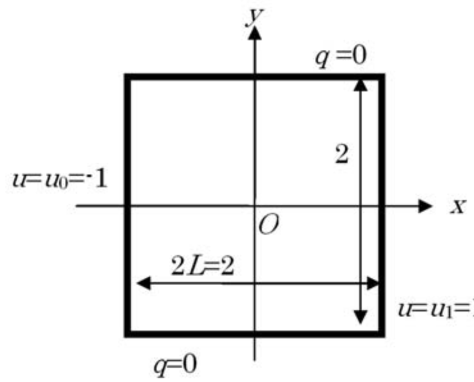

Fig. 1: Object under Consideration

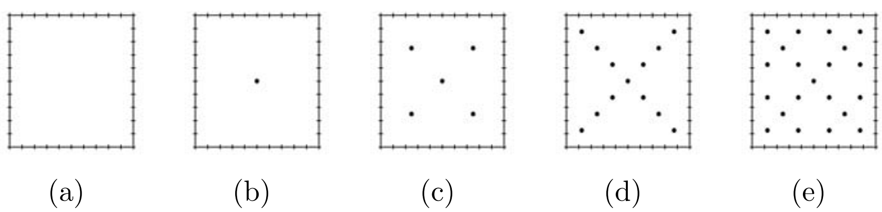

Fig. 2: Placement of computing points

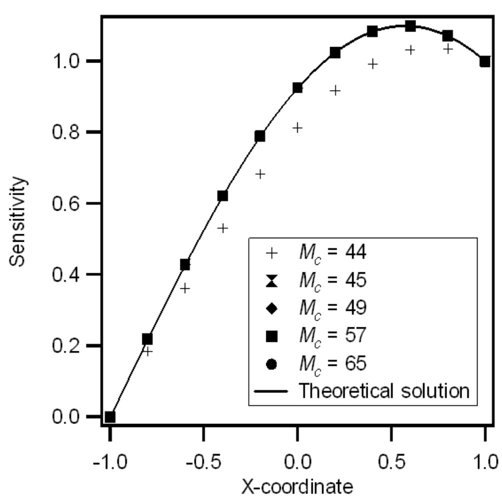

Fig. 3: Distribution of sensitivity with respect to specified value

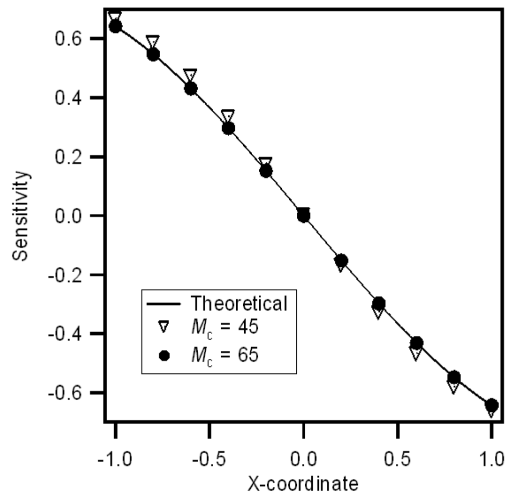

Fig. 4: Distribution of sensitivity with respect to shape parameter

44 collocation points are placed uniformly on the boundary. Two collocation points, which are placed at each corner point, have the same coordinates and different normal vector. In addition to the boundary collocation points, some internal points are taken as computing points. The placement of the points are shown in Fig.2.

First, we shall take specified value u_1 as the variable for the sensitivity analysis. Distributions of numerical and theoretical solutions of sensitivities are compared in Fig.3. The abscissa and the ordinate denote x-coordinates of the collocation points and numerical solutions of sensitivities, respectively. We notice that the difference between numerical and theoretical solutions is improved rapidly as the number of computing points increases.

Next, we shall take specified value L as the variable for the sensitivity analysis. Distributions of numerical solutions for sensitivities in case of $M_c = 45$ and 65 are compared in Fig.4. We notice that the difference between numerical and theoretical solutions in case of $M_c = 45$ is relatively greater than that in case of $M_c = 65$.

Reference

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