Stress analysis for a porous medium containing a cylindrical rigid inclusion

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Summary

In the present work a plane strain problem of the equilibrium theory of elastic materials with voids is studied. The problem of a rigid inclusion in an infinite body is investigated. The solution generalize analogous results in classical elasticity.

Introduction

This paper concerns plane problems in the equilibrium theory of linear elastic materials with voids. This theory was formulated by Cowin and Nunziato [1] as a linearization of a nonlinear theory for elastic porous bodies. The linear theory deals with small changes from a reference configuration of porous body. The independent kinematic variables are the displacement field u_i and the change in volume fraction ψ.

The intended application of the theory is to behavior of solid materials with small, distributed voids as geological materials and biological materials.

In this paper we study the problem of a rigid inclusion in an infinite body which is uniformly stretched along one axis. This problem is of great practical and technological importance and in the context of classical elasticity has been a subject of various studies (see, e.g. [2,3]). In Section 2 we present the basic equations of the equilibrium theory of elastic materials with voids and derive the equations of the plane strain problem for homogeneous and isotropic bodies. Section 3 concerns the problem of a cylindrical rigid inclusion. The solution is presented in a closed form and generalize analogous results in classical elasticity.

Basic Equations

Throughout this section B is a regular region of three-dimensional Euclidean space. We let ∂B denote the boundary of B and designate by n the outward unit normal of ∂B. We assume that the region B is occupied by a linearly elastic material with voids. The body is referred to a system of rectangular Cartesian axes Ox_i. Let u be the displacement field over B. The linear strain measure e_{ij} is given by

$$e_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right). \tag{1}$$

Let t_{ij} be the stress tensor and let h_i be the equilibrated stress vector. The components of surface traction t_i and the equilibrated stress h are given by

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$$t_i = t_{ji} n_j, \qquad h = h_i n_i, \tag{2}$$

respectively. The equilibrium equations are

$$t_{ji,j} + f_i = 0, \qquad h_{i,i} - g + l = 0, \tag{3}$$

where f_i are the components of body force, g is the intrinsic equilibrated body force and l is the extrinsic equilibrated body force.

In the case of centrosymmetric isotropic material the constitutive equations are

$$t_{ij} = \lambda e_{rr} \delta_{ij} + 2\mu e_{ij} + \beta \psi \delta_{ij}, \qquad h_i = \alpha \psi,_i, \qquad g = \beta e_{rr} + \zeta \psi, \tag{4}$$

where ψ is the volume fraction function, δ_{ij} is Kronecker's delta, and $\lambda,\ \mu,\ \beta,\ \alpha$ and ζ are constitutive coefficients. We restrict our attention to homogeneous materials so that the constitutive coefficients are constants. We assume that the internal energy density is a positive definite form. This assumption implies that [1]

$$\mu > 0,\ \alpha > 0,\quad \zeta > 0,\ 2\mu + 3\lambda > 0,\quad \left(2\mu + 3\lambda\right)\zeta > 3b^2. \tag{5}$$

We assume that the region B refers to a right cylinder with the open cross section Σ and the smooth lateral boundary Π. The rectangular Cartesian coordinate frame is supposed to be chosen in such a way that the $x_3 - axis$ is parallel to the generators of B.We denote by L the boundary of Σ.

In what follows we are interested in a plane strain problem with the displacement vector and the volume fraction function being specified in cylindrical coordinates (r,θ,z) as follows:

$$u_r = u(r,\theta),\quad u_\theta = v(r,\theta),\quad u_z = 0,\quad \psi = \varphi(r,\theta),\ (r,\theta) \in \Theta. \tag{6}$$

The geometrical equations (1) become

$$\varepsilon_{rr} = \frac{\partial u}{\partial r},\quad \varepsilon_{\theta\theta} = \frac{1}{r}\left(\frac{\partial v}{\partial \theta} + u\right),\quad \varepsilon_{r\theta} = \frac{1}{2}\left(\frac{1}{r}\frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial r} - \frac{1}{r}v\right). \tag{7}$$

The equilibrium equations (2) take the form

$$\begin{aligned}
&\frac{\partial \tau_{rr}}{\partial r} + \frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{1}{r}\left(\tau_{rr} - \tau_{\theta\theta}\right) = 0,\\
&\frac{\partial \tau_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \tau_{\theta\theta}}{\partial \theta} + \frac{2}{r}\tau_{r\theta} = 0,\\
&\frac{1}{r}\frac{\partial}{\partial r}\left(r\chi_r\right) + \frac{1}{r}\frac{\partial \chi_\theta}{\partial \theta} - \gamma = 0.
\end{aligned} \tag{8}$$

The constitutive equations (4) can be written in the form

$$\begin{aligned}
&\tau_{rr} = \left(\lambda + 2\mu\right)\varepsilon_{rr} + \lambda\varepsilon_{\theta\theta} + \beta\varphi,\quad \tau_{\theta\theta} = \lambda\varepsilon_{rr} + \left(\lambda + 2\mu\right)\varepsilon_{\theta\theta} + \beta\varphi,\quad \tau_{r\theta} = 2\mu\varepsilon_{r\theta},\\
&\chi_r = \alpha\frac{\partial \varphi}{\partial r},\quad \chi_\theta = \frac{1}{r}\alpha\frac{\partial \varphi}{\partial \theta},\quad \gamma = \beta\left[\frac{1}{r}\frac{\partial}{\partial r}\left(ru\right) + \frac{1}{r}\frac{\partial v}{\partial \theta}\right] + \zeta\varphi.
\end{aligned} \tag{9}$$

The plane strain problem consists in the finding of the functions $u,\ v$ and ψ on Σ, which satisfy the Eqs.(7)-(9) and the boundary conditions.

The problem of a rigid inclusion

In this section we study the problem of a rigid cylindrical inclusion in an infinite body which is uniformly stretched along the axis Ox_1. We assume that the elastic body occupies the region $B=\left\{(x_1,x_2,x_3)\in R^3 : x_1^2+x_2^2>a^2\right\}$, where a is a positive constant. We assume that the region $\left\{(x_1,x_2,x_3)\in R^3 : x_1^2+x_2^2<a^2\right\}$ is occupied by a rigid body. We consider the following boundary conditions:

$$u_r=0,\quad u_\theta=0,\quad \varphi=0, \text{ on } r=a, \tag{10}$$

and the conditions at infinity

$$\tau_{rr}=\frac{1}{2}P\left(1+\cos 2\theta\right),\quad \tau_{\theta\theta}=\frac{1}{2}P\left(1-\cos 2\theta\right),\quad \tau_{r\theta}=\tau_{\theta r}=-\frac{1}{2}P\sin 2\theta,\quad \chi_r=\chi_\theta=0 \tag{11}$$

where P is a given constant. the body B is in a state of plane strain parallel to the plane x_1Ox_2 in the absents of body loads. We seek the solution in the form

$$u=F(r)+U(r)\cos 2\theta,\quad v=V(r)\sin 2\theta,\quad \varphi=G(r)+\Phi(r)\cos 2\theta, \tag{12}$$

where F, G, U, V and Φ are function only on r. It follows from (7), (12) and (9) that

$$\begin{gathered}
\tau_{rr}=(\lambda+2\mu)F'+\frac{1}{r}\lambda F+\beta G+\left[(\lambda+2\mu)U'+\frac{1}{r}\lambda(U+2V)+\beta\Phi\right]\cos 2\theta,\\
\tau_{\theta\theta}=\lambda F'+\frac{1}{r}(\lambda+2\mu)F+\beta G+\left[\lambda U'+\frac{1}{r}(\lambda+2\mu)(U+2V)+\beta\Phi\right]\cos 2\theta,\\
\tau_{r\theta}=\tau_{\theta r}=\left[\mu V'-\frac{1}{r}\mu(2U+V)\right]\sin 2\theta,\quad \chi_r=\alpha\left(G'+\Phi'\cos 2\theta\right),\quad \chi_\theta=-\frac{2}{r}\alpha\sin 2\theta,\\
\gamma=\beta\left(F'+\frac{1}{r}F\right)+\zeta G+\left\{\beta\left[U'+\frac{1}{2}(U+2V)\right]+\zeta\Phi\right\}\cos 2\theta,
\end{gathered} \tag{13}$$

where the prime denote derivation respect to r .If we substitute (13) in the equilibrium equations (8), we obtain the following equations:

$$\begin{gathered}
(\lambda+2\mu)\left(F''+\frac{1}{r}F'-\frac{1}{r^2}F\right)+\beta G'=0,\quad \alpha\left(G''+\frac{1}{r}G'-\frac{\zeta}{\alpha}G\right)-\beta\left(F'+\frac{1}{r}F\right)=0,\\
(\lambda+2\mu)\left(r^2U''+rU'\right)+(\mu+\lambda)rV'+\beta r^2\Phi-(\lambda+6\mu)U-2(\lambda+3\mu)V=0,\\
\mu\left(r^2V''+rV'\right)-2(\lambda+2\mu)rU'-2(\lambda+3\mu)U-(4\lambda+9\mu)V-2\beta r\Phi=0,\\
\alpha\left(r^2\Phi''+r\Phi'-4\Phi-\frac{\zeta}{\alpha}r^2\Phi\right)-\beta r^2U'-\beta r(U+2V)=0.
\end{gathered} \tag{14}$$

The first equation of (14) implies that

$$F'+\frac{1}{r}F+\frac{\beta}{(2\mu+\lambda)}G=C_1, \tag{15}$$

where C_1 is an arbitrary constant. In view of (15), the second equation of (14) can be written in the form

$$G'' + \frac{1}{r}G' - \xi^2 G = \frac{\beta}{\alpha}C_1, \tag{16}$$

where

$$\xi^2 = \frac{1}{\alpha}\left(\zeta - \frac{\beta^2}{\lambda + 2\mu}\right). \tag{17}$$

It follows from (5) that $\xi^2 > 0$. Since the function G must be finite at infinity, the solution of Eq. (16) is

$$G = A_1 K_0\left(\xi r\right) - \frac{\beta}{\xi^2 \alpha}C_1. \tag{18}$$

where I_n and K_n are the modified Bessel functions of order n, A_1 is an arbitrary constant. It follows from (18) and (15) that

$$F = \frac{\zeta}{2\xi^2\alpha}C_1 r + \frac{1}{r}C_2 + \frac{\beta A_1}{\xi\left(\lambda + 2\mu\right)}K_1\left(\xi r\right), \tag{19}$$

where C_2 is an arbitrary constant. Now we introduce the independent variable t through the relation $t = \ln r$, and denote $D = d/dt$. Then, Eqs.$(14)_{3,4}$ can be written in the form

$$\begin{aligned} \left[D^2 - \left(1 + 4c_1\right)\right]U + 2\left[\left(1 - c_1\right)D - \left(1 + c_1\right)\right]V &= -e^t c_2 D\Phi, \\ \left[\left(1 - c_1\right)D + \left(1 + c_1\right)\right]U + \left[c_1 D^2 - \left(4 + c_1\right)\right]V &= 2e^t c_2 \Phi, \end{aligned} \tag{20}$$

where

$$c_1 = \frac{\mu}{\lambda + 2\mu}, \quad c_2 = \frac{\beta}{\lambda + 2\mu}. \tag{21}$$

The general solution of the homogeneous system (20) which corresponds to a finite stress field at infinity is given by

$$U_0 = b_1 e^{-t} + B_2 e^{-3t} + B_3 e^t, \quad V_0 = -c_1 B_1 e^{-t} + B_2 e^{-3t} - B_3 e^t, \tag{22}$$

where B_1, B_2 and B_3 are arbitrary constants. Particular solution of the system (20) can be seen to be

$$U^* = -\frac{1}{2}c_2\left(e^t S_1 + e^{-3t}S_2\right), \quad V^* = \frac{1}{2}c_2\left(e^t S_1 - e^{-3t}S_2\right), \tag{23}$$

where

$$S_1\left(t\right) = \int_t \Phi\left(s\right)ds, \quad S_2\left(t\right) = \int_t e^{4s}\Phi\left(s\right)ds. \tag{24}$$

With the help of (22) and (23) we obtain

$$U = B_1 r^{-1} + B_2 r^{-3} + B_3 r - \frac{1}{2}c_2\left[r\int_r x^{-1}\Phi\left(x\right)dx + r^{-3}\int_r x^3\Phi\left(x\right)dx\right],$$

$$V = -c_1 B_1 r^{-1} + B_2 r^{-3} - B_3 r + \frac{1}{2} c_2 \left[r \int_r x^{-1} \Phi(x) dx - r^{-3} \int_r x^3 \Phi(x) dx \right]. \quad (25)$$

If we substitute U and V from (25) we obtain the equation

$$r^2 \Phi'' + r\Phi' - \left(4 + \xi^2 r^2\right) \Phi = -\frac{2c_1 \beta B_1}{\alpha}. \quad (26)$$

The solution of Eq.(26) which generate finite stresses for $r \to \infty$ are given by

$$\Phi = A_2 K_2(\xi r) + \frac{2}{\xi^2 \alpha} c_1 \beta B_1 r^{-2}, \quad (27)$$

where A_2 is an arbitrary constant. If we substitute (27) into relations (25) we obtain

$$\begin{aligned} U &= \frac{1}{r} B_1 + \frac{1}{r^3} B_2 + B_3 r + \frac{1}{2\xi} c_2 A_2 \left[K_3(\xi r) - K_1(\xi r) \right], \\ V &= -\frac{1}{r} c_2 d B_1 + \frac{1}{r^3} B_2 - B_3 r + \frac{1}{2\xi} c_2 A_2 \left[K_3(\xi r) - K_1(\xi r) \right], \end{aligned} \quad (28)$$

where $d = \zeta / \alpha \xi^2$. We introduce the notations

$$q_1 = 1 - 2c_1 d, q_2 = 2 - c_1 d, \quad Q = \frac{\zeta(\lambda + \mu) - \beta^2}{\zeta(\lambda + \mu) - \beta^2}, \quad K = 2\zeta(\lambda + \mu) - \beta^2, \quad (29)$$

It follows from (9),(18),(19) and (27)-(29) that

$$\begin{aligned} t_{rr} &= \frac{K}{2\alpha\xi^2} c_1 - 2\mu r^{-2} c_2 - 2\mu \Big\{ 2Q r^{-2} B_1 + 3r^{-4} B_2 - B_3 + \frac{1}{4\xi} c_2 A_3 \big[6r^{-1} K_3(\xi r) - \\ &\quad \xi K_2(\xi r) + \xi K_0(\xi r) \big] \Big\} \cos 2\theta, \\ t_{\theta\theta} &= \frac{K}{2\alpha\xi^2} c_1 + 2\mu r^{-2} c_2 + 2\mu c_2 A_1 \left[K_0(\xi r) + \frac{1}{\xi r} K_1(\xi r) \right] + 2\mu \Big\{ 3r^{-4} B_2 - B_3 + \\ &\quad \frac{1}{4\xi} c_2 A_3 \big[3\xi K_2(\xi r) + \xi K_0(\xi r) - 6r^{-1} \xi K_3(\xi r) \big] \Big\} \cos 2\theta, \\ t_{r\theta} &= 2\mu \left\{ -\frac{Q}{2} B_1 r^{-2} - 3r^{-4} B_2 - B_3 - \frac{1}{2\xi} c_2 A_3 r^{-1} \left[3K_3(\xi r) + K_1(\xi r) \right] \right\} \sin 2\theta, \\ h_r &= -\alpha\xi A_1 K_1(\xi r) - \left\{ \alpha A_3 \left[\xi K_1(\xi r) + 2r^{-1} K_1(\xi r) \right] + 4B_1 r^{-3} c_1 \frac{\beta}{\xi^2} \right\} \cos 2\theta, \\ h_\theta &= -\left\{ 2\alpha r^{-1} A_3 K_3(\xi r) + 4B_1 r^{-3} c_1 \frac{\beta}{\xi^2} \right\} \sin 2\theta. \end{aligned} \quad (30)$$

On the basis of (30) the conditions at infinity (11) reduce to

$$B_3 = \frac{1}{4\mu} P, \quad C_1 = \frac{\alpha\xi^2}{K} P. \quad (31)$$

We note that the restrictions (5) imply that $K>0$. With help of (18), (19),(27) and (28) we find that the conditions (6) can be written in the form

$$A_1=\frac{\beta}{K}P\left[K_0(\xi a)\right]^{-1},\ C_2=-\frac{\zeta a^2}{2K}P-\frac{c_2 a}{\xi}A_1K_1(\xi a),\ A_2=-\frac{2}{\alpha\xi^2a^2}c_1\beta B_1\left[K_2(\xi a)\right]^{-1},$$
$$\left[a^2-\frac{c_1c_2\beta a}{a\xi^3}L(\xi a)\right]B_1+B_2=-\frac{Pa^4}{4\mu},\quad \left[-c_2da^2-\frac{c_1c_2\beta a}{a\xi^3}L(\xi a)\right]B_1+B_2=\frac{Pa^4}{4\mu}, \tag{32}$$

where

$$L(z)=\left[K_3(z)+K_1(z)\right]\left[K_2(z)\right]^{-1}. \tag{33}$$

From (32) we obtain

$$B_1=-\frac{Pa^2}{2\mu(1+c_2d)},\quad B_2=\frac{Pa^2}{4\mu(1+c_2d)}\left[(1-c_2d)a^2-\frac{2c_1c_2\beta a}{a\xi^3}L(\xi a)\right]. \tag{34}$$

The solution of the problem has the form (12) where the constants A_α, B_i and C_α are given by (31), (32) and (34).The stress tensor and microstress vector can be determined from the relations (30).In particular, the values of t_{rr} and $t_{r\theta}$ on the boundary of the inclusion have the form

$$t_{rr}=\frac{1}{2}P+\frac{\mu}{K}\zeta P\left[\frac{2\beta c_2}{\xi\zeta a}\frac{K_1(\xi a)}{K_0(\xi a)}+1\right]-\frac{Pa^2}{1+c_2d}\left[(-2Q+1-2c_2d)a^2-\frac{4c_1c_2\beta K_1(\xi a)}{\alpha\xi^3a^3K_2(\xi a)}\right]\cos 2\theta,$$
$$t_{r\theta}=-\frac{Pa^2}{1+c_2d}\left[\left(-Q+\frac{1}{2}-\frac{5}{2}c_2d\right)a^{-2}-\frac{2c_1c_2\beta K_1(\xi a)}{\alpha\xi^2a^3K_2(\xi a)}\right]\sin 2\theta. \tag{35}$$

The problem of a rigid inclusion in an elastic medium has been investigated also in the context of non-classical theories (see e.g. [4-5]).

Reference

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