Wave Propagation Analysis in FGMs

Ch. Zhang[1], J. Sladek[2], V. Sladek[2], S. Hirose[3]

Summary

Wave propagation in functionally graded materials (FGMs) is analyzed. To gain a better understanding of the wave propagation characteristics in FGMs, a one-dimensional model is first considered. Exponential laws are used to describe the spatial variation of the Young's modulus and the mass density of the FGMs. Numerical examples are given to show the effects of the material gradation on the shape and the amplitude of the stress waves. Numerical methods for two- and three-dimensional wave propagation simulations are briefly discussed.

Introduction

Functionally graded materials (FGMs) are a new class of composites designed to achieve high performance levels superior to homogeneous materials by combining the desirable properties of each constituent. In ideal case, FGMs have a continuous material profile graded from one constituent to another. In this way, desirable advantages of both constituents can be incorporated. FGMs have superior in-service advantages to meet the increasingly growing demands on engineering materials to withstand extreme loading conditions, such as super-high temperatures and mechanical impacts.

In many real situations, FGMs may be subjected to a mechanical impact loading, which initiates elastic waves propagating through the solids. Thus, an important issue in the research of FGMs is the wave propagation analysis in such materials. The corresponding results have direct relevant applications in the design, optimization, safety and structural integrity analysis, as well as ultrasonic non-destructive testing of FGMs. With this motivation in mind, wave propagation analysis is presented in this paper. First we present a simple one-dimensional model, and then we give a brief review on numerical methods for two- and three-dimensional wave propagation simulations.

One-dimensional Wave Propagation

To gain a deeper insight into the wave propagation characteristics in FGMs, let us first consider one-dimensional wave propagation in a FGM rod. The rod has a length l and is fixed on one end at $x = 0$, while an impact load $f(t)$ is applied on another end of the rod at $x = l$. The spatial variations of the Young's modulus and the mass density are described by the following exponential laws

$$E(x) = E_0 e^{\alpha x}, \qquad \rho(x) = \rho_0 e^{\beta x}, \tag{1}$$

[1]Dept. Civil Eng., University of Applied Sciences Zittau/Görlitz, D-02763 Zittau, Germany
[2]Institute of Construction and Architecture, Slovak Academy of Sciences, 84503 Bratislava, Slovakia
[3]Dept. Civil Eng., Tokyo Institute of Technology, 2-12-1, O-okayama, Meguro-ku, Tokyo, 152-8552, Japan

where E_0 and ρ_0 are the material constants at $x = 0$, while α and β are the gradient parameters, which can be obtained by using the following relations

$$\alpha = \frac{1}{l}\log\left[\frac{E(l)}{E_0}\right], \qquad \beta = \frac{1}{l}\log\left[\frac{\rho(l)}{\rho_0}\right]. \tag{2}$$

The equation of motion is given by

$$\frac{\partial\sigma(x,t)}{\partial x} = \rho(x)\frac{\partial^2 u(x,t)}{\partial^2 t}. \tag{3}$$

Substitution of the generalized Hooke's law

$$\sigma(x,t) = E(x)\frac{\partial u(x,t)}{\partial x} \tag{4}$$

into Eq. (3) we obtain the following one-dimensional wave equation

$$\frac{\partial}{\partial x}\left[E(x)\frac{\partial u(x,t)}{\partial x}\right] = \rho(x)\frac{\partial^2 u(x,t)}{\partial^2 t}. \tag{5}$$

Substitution of Eq. (1) into Eq. (5) yields

$$\alpha\frac{\partial u(x,t)}{\partial x} + \frac{\partial^2 u(x,t)}{\partial^2 x} = \frac{1}{c_0^2}e^{-(\alpha-\beta)x}\frac{\partial^2 u(x,t)}{\partial^2 t}, \tag{6}$$

in which $c_0 = \sqrt{E_0/\rho_0}$ is the wave velocity at $x = 0$.

As initial and boundary conditions we have

$$u(x,0) = \dot{u}(x,0) = 0, \quad \text{for} \quad t = 0, \tag{7}$$

$$u(0,t) = 0, \quad \text{at} \quad x = 0; \qquad \sigma(l,t) = f(t), \quad \text{at} \quad x = l, \tag{8}$$

By applying the Laplace-transform to Eq. (6) we obtain

$$\alpha\frac{\partial\overline{u}(x)}{\partial x} + \frac{\partial^2\overline{u}(x)}{\partial^2 x} = \frac{p^2}{c_0^2}e^{-(\alpha-\beta)x}\overline{u}(x), \tag{9}$$

where p is the Laplace-transform parameter and an over-bar denotes the Laplace-transform of a quantity. The transformed boundary conditions can be written as

$$\bar{u}(0) = 0\,, \quad \text{at} \quad x = 0\,; \qquad \bar{\sigma}(l) = \bar{f}(p)\,, \quad \text{at} \quad x = l\,, \tag{10}$$

Equation (9) in conjunction with Hooke's law (4) has the following solutions

$$\bar{u}(x) = e^{-\frac{1}{2}\alpha x}\left[c_1 J_\nu(z) + c_2 Y_\nu(z)\right]\,, \tag{11}$$

$$\bar{\sigma}(x) = -\frac{1}{2}E_0(\alpha-\beta)z\left[c_1 J_{\nu-1}(z) + c_2 Y_{\nu-1}(z)\right]\,, \tag{12}$$

where $J_\nu(z)$ and $Y_\nu(z)$ are the ν-th order Bessel functions of the first and second kind with

$$\nu = \frac{\alpha}{\alpha-\beta}\,, \qquad z = 2i\left|\frac{p}{\alpha-\beta}\right| e^{-\frac{1}{2}(\alpha-\beta)x}\,. \tag{13}$$

It should be remarked here that $\alpha = \beta = 0$ corresponds to the homogeneous case. If we assume the same exponential spatial variation for the Young's modulus and the mass density, i.e., $\alpha = \beta \neq 0$, then a constant wave velocity throughout the graded rod is obtained. This case has been often considered in literature to simplify the analysis and make the investigation tractable. The general case $\alpha \neq \beta \neq= 0$, however, has yet not been investigated in literature to the best of the author's knowledge.

The constants c_1 and c_2 in Eqs. (11) and (12) can be obtained by using the transformed boundary conditions (10), but they are not given here for the sake of brevity.

After the solutions have been obtained for discrete Laplace-transform parameters, the corresponding time-dependent solutions can be obtained by using an inverse Laplace-transform algorithm. In this analysis, we apply the automatic inversion algorithm of D'Amore *et al.* [1]

As external loading, we consider an impact pressure loading of the form

$$f(t) = -\sigma_0\left[H(t) - H(t-t_0)\right]\,. \tag{14}$$

Equation (14) represents a rectangular pulse with an amplitude σ_0 and a duration t_0. Numerical calculations are carried out for a pulse duration of $t_0 = 0.2l/c_0$.

For a homogeneous rod, the normalized stress at $x = 0$ versus the dimensionless time is shown in Fig. 1. The numerical results agree very well with the analytical one. Characteristic for wave propagation in a homogeneous rod are: 1) The pulse shape remains unchanged,

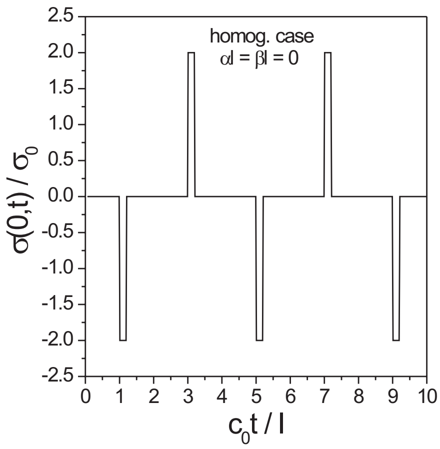

Figure 1: Normalized stress at $x = 0$

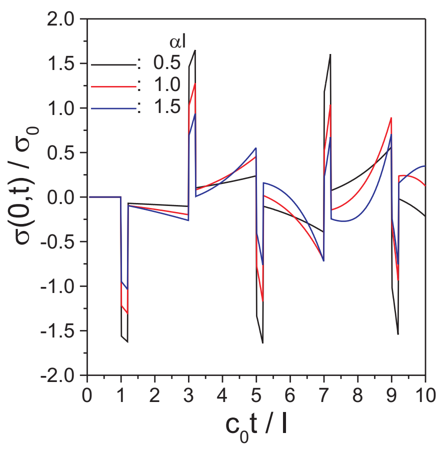

Figure 2: Normalized stress at $x = 0$

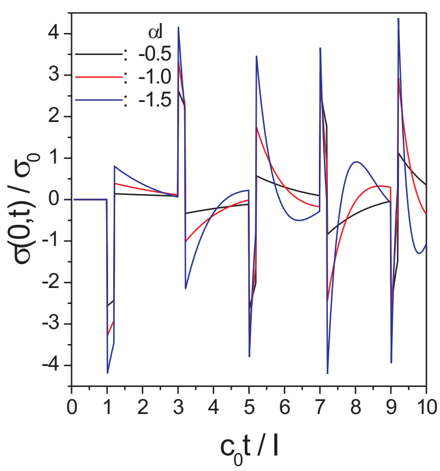

Figure 3: Normalized stress at $x = 0$

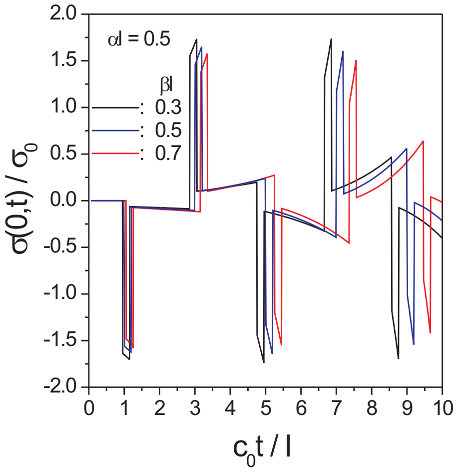

Figure 4: Normalized stress at $x = 0$

i.e., without pulse shape distortion, and 2.) the pulse amplitude remains constant. As is well known, the stress amplification factor for a homogeneous rod is 2.

Figures 2 and 3 show the time-dependence of the normalized stress at $x = 0$ for a FGM rod with constant wave velocity, i.e., $\alpha l = \beta l \neq 0$. In comparison to the homogeneous case, there are two distinct differences: 1.) The material gradation gives rise to a pulse distortion, and 2.) there is a change in the pulse amplitude. Depending on the material gradation, the pulse amplitude may be amplified or reduced.

For the general case with $\alpha l \neq \beta l \neq 0$, the normalized stress at $x = 0$ is shown in Fig. 4. As in the case of $\alpha l = \beta l \neq 0$, the pulse shape is now distorted, and the pulse amplitude is changed. Compared to the case with $\alpha l = \beta l \neq 0$, no qualitative differences are observed. However, there are quantitative differences between both cases, both in the pulse amplitude and in the time, at which the maximum pulse amplitude is reached. Qualitatively, our numerical results agree very well with those of Chiu and Erdogan [2], who investigated a similar problem by using power-laws to describe the material gradients.

Two- and Three-dimensional Wave Propagation

Two- and three-dimensional wave propagation analysis requires sophisticated numerical methods due to the mathematical complexity of the governing partial differential equations. For this purpose, the following methods can be applied:

- Multi-layer Model ([3], [4]);
- Finite Element Method (FEM) ([5]-[7]);
- Finite Volume Method (FVM) ([8], [9]);
- Boundary Element Method (BEM) [10].

In the multi-layer model ([3], [4]), the analyzed domain is divided into several strips or layers along the direction of the material gradient. In each layer, the material properties are assumed to be constant, or linear, or quadratic. The essential disadvantage of this method is that it cannot be easily extended to FGMs with bi- or tri-directional material gradations. In addition, the method may not be suitable for cases with complicated geometrical boundaries, in which the material gradation is neither perpendicular nor parallel to the boundaries of the FGMs.

The FEM is the most widely used and thoroughly developed computational method. However, the most commercial FEM codes assume constant material properties within each element, which may not be appropriate for nonhomogeneous solids such as FGMs. To approximate the spatial variation of the material constants reasonably, very fine meshes have to be applied, which increase the computational effort and reduces the efficiency of the FEM ([5], [6]). An improvement of the classical FEM can be achieved by using the so-called graded finite elements [7], which allow a spatial variation of the material parameters within each element.

The FVM ([8], [9]) has the same trouble in dealing with the spatial variation of the material properties within each element. This method is beneficial for FGMs with rapidly varying material properties. Unfortunately, this method is yet less known in literature and not widely applied.

The BEM is an efficient and popular alternative to the FEM in wave propagation analysis. Though the BEM has been successfully applied to wave propagation simulations in homogeneous materials since many years, its application to FGMs is unfortunately yet very limited due the fact that the corresponding fundamental solutions or Green's functions for general FGMs are either not available or mathematically too complex. The nonhomogeneous nature of FGMs prohibits an easy construction of Green's functions for general cases. To overcome this difficulty, a meshless local BEM or local boundary integral equation method (LBIEM) based on the meshless local Petrov-Galerkin (MLPG) method has been recently developed in [10] for elastodynamic analysis of FGMs. This method extends the applicability range of the classical BEM to continuously nonhomogeneous solids, and has certain advantages in comparison to other numerical methods.

Wave propagation in FGMs is an interesting and important research area, which needs further intensive analytical, numerical and experimental investigations.

Reference

1. D'Amore, L., Laccetti, G. and Murli, A. (1999): "An Implementation of a Fourier Series Method for the Numerical Inversion of the Laplace Transform", *ACM Transactions on Mathematical Software*, Vol. 25, pp. 279-305.
2. Chiu, T.-C. and Erdogan, F. (1999): "One-dimensional Wave Propagation in a Functionally Graded Elastic Medium", *Journal of Sound and Vibration*, Vol. 222, pp. 453-487.
3. Han, X., Liu, G.R. and Lam, K.Y. (2001): "Transient Waves in Plates of Functionally Graded Materials", *Int. J. Numer. Methods Engrg.*, Vol. 52, pp. 851-865.
4. Han, X., Liu, G.R., Xi, Z.C. and Lam, K.Y. (2001): "Transient Waves in a Functionally Graded Cylinder", *Int. J. Solids Struct.*, Vol. 38, pp. 3021-3037.
5. Li, Y., Ramesh, K.T. and Chin, E.S.C. (2001): "Dynamic Characterization of Layered and Graded Structures Under Impulsive Loading", *Int. J. Solids Struct.*, Vol. 38, pp. 6045-6061.
6. Banks-Sills, L., Eliasi, R. and Berlin, Y. (2002): "Modeling of Functionally Graded Materials in Dynamic Analyses", *Composites Part B*, Vol. 33, pp. 7-15.
7. Santare, M.H., Thamburaj, P. and Gazonas G.A. (2003): "The Use of Graded Finite Elements in the Study of Elastic Wave Propagation in Continuously Nonhomogeneous Materials", *Int. J. Solids Struct.*, Vol. 40, pp. 5621-5634.
8. Fogarty, T.R. and LeVeque, R.J. (1999): "High-resolution Finite Volume Methods for Acoustic Waves in Periodic and Random Media", *J. Acoust. Soc. Amer.*, Vol. 106, pp. 17-28.
9. Berezovski, A., Engelbrecht, J. and Maugin, G.A. (2003): "Numerical Simulation of Two-dimensional Wave Propagation in Functionally Graded Materials", *European Journal of Mechanics A/Solids*, Vol. 22, pp. 257-265.
10. Sladek, J., Sladek, V. and Zhang, Ch. (2003): "Application of Meshless Local Petrov-Galerkin (MLPG) Method to Elastodynamic Problems in Continuously Nonhomogeneous Solids", *Computer Modeling in Engineering & Sciences*, Vol. 4, pp. 637-647.