Instabilities and Loss of Ellipticity in Fiber-Reinforced Nonlinearly Elastic Solids under Plane Deformation

J. Merodio[1], R. W. Ogden[2]

Summary

Material instabilities for fiber-reinforced nonlinearly elastic solids are examined under plane deformation. The materials under consideration are isotropic nonlinear elastic models augmented with a function that accounts for the existence of unidirectional reinforcing (the so-called *reinforcing model*). The onset for failure is associated with the loss of ellipticity of the governing differential equations. Previous work has dealt with the analysis of specific reinforcing models and it has been established that the loss of ellipticity for the augmented isotropic materials concerned requires contraction in the reinforcing direction. The loss of ellipticity was related to fiber kinking. Here we describe generalizations of these results in respect of both compressible and incompressible materials. The incipient loss of ellipticity is interpreted in terms of fiber kinking, fiber de-bonding, fiber splitting and matrix failure in fiber-reinforced composite materials.

Introduction

Failure mechanisms in unidirectionally reinforced composite materials have received increased attention in the last few years. These failure mechanisms include fiber kinking [1–4], fiber splitting [5], fiber de-bonding [6] and matrix failure [7] in particular. The analyses in these contributions provide different theories that capture and explain the failure modes for the materials under consideration. However, a unified derivation that enables prediction of fiber instability or fiber failure in fiber-reinforced composite materials has only recently been established, for incompressible [8] and compressible materials [9], respectively.

Many factors affect the mechanical interaction between the fibers and the matrix. In this paper, we focus on a continuum-mechanical model based on nonlinear elasticity that addresses the material instabilities mentioned, specifically fiber kinking, fiber splitting, fiber de-bonding and matrix failure. The onset for failure is signalled by the loss of ellipticity of the governing differential equations [2–4]. The goal is to capture and predict the failure mechanisms associated with particular fiber-reinforced materials. A sufficiently general strain energy depending on deformation invariants that penalize deformation in a particular direction serves as the material model. The loss of ellipticity condition enables both the deformation associated with the existence of surfaces of weak discontinuity and the normal to that surface to be identified for any particular strain-energy function. Surfaces of weak discontinuity (or weak surfaces) are surfaces across which at least one of the second derivatives of the displacement field is discontinuous, whereas a strong surface of discontinuity is one across which the displacement gradient is discontinuous.

[1]Department of Structural and Mechanical Engineering, University of Cantabria, Santander, Spain
[2]Department of Mathematics, University of Glasgow, Glasgow G12 8QW, UK

In the present analysis, for a given strain energy, we relate the angle between (the normal to) a weak surface and the direction of the fiber reinforcement to a particular failure mechanism. The argument is as follows. Under fiber contraction the onset of fiber kinking is associated with a weak surfaces close to the normal to the direction of fiber reinforcement [1]. Thus, if loss of ellipticity is associated with a weak surface perpendicular to the fiber under fiber contraction, then the associated fiber failure mechanism is identified as fiber kinking. On the other hand, for fiber de-bonding, the angle between the weak surface and the fiber reinforcement direction is close to zero [6]. For fiber kinking combined with fiber splitting, the simultaneous existence of weak surfaces close to and normal to the fiber direction is required [5]. In fiber extension, matrix failure is associated with weak surfaces perpendicular to the direction of fiber reinforcement [7].

Constitutive equations that suffer a loss of ellipticity have been studied in a variety of contexts [2–4]. In particular, [3] and [4] deal with the loss of ellipticity of some particular transversely isotropic nonlinearly elastic materials under plane deformation. In these analyses an isotropic base material is augmented with a uniaxial reinforcement in the fiber direction, which lies within the considered plane of deformation. Here, following the general analysis in [8, 9] we adopt this approach and define the strain energy in terms of an isotropic base augmented with a reinforcing model.

In three dimensions, two invariants are sufficient to characterize the anisotropic nature of a transversely isotropic material model. These invariants are deformation measures, one, denoted I_4, being related only to the stretch in the direction of the fiber reinforcement. The so-called *standard reinforcing model* is a quadratic function that depends only on this invariant. The second invariant, denoted I_5, also responds to changes in the stretch in the fiber direction but introduces an additional effect that may be associated with the behavior of the reinforcement under shear deformations. For plane deformations with the fiber direction lying in the considered plane the invariants are not independent. Nevertheless, we consider the influence of each invariant separately since each invariant adds a distinct anisotropic character to the isotropic base material.

The Material Model and Ellipticity

For homogeneous transversely isotropic nonlinear elastic solids the most general strain-energy function (measured per unit undeformed volume) may be expressed in the form $W = W(I_1, I_2, I_3, I_4, I_5)$ (see Spencer [10]). Here I_1, I_2, I_3 are the principal isotropic invariants of the Cauchy-Green strain tensor $\mathbf{C} = \mathbf{F}^T\mathbf{F}$, where $\mathbf{F} = \partial\mathbf{x}/\partial\mathbf{X}$ is the deformation gradient tensor defined on the region occupied by the body in its undeformed configuration, $\mathbf{X}$ being the position vector of a particle in the undeformed configuration and $\mathbf{x}$ the corresponding position vector in the deformed configuration. These invariants are given by

$$I_1 = \mathrm{tr}\,\mathbf{C}, \quad I_2 = \frac{1}{2}[(\mathrm{tr}\,\mathbf{C})^2 - \mathrm{tr}\,(\mathbf{C}^2)], \quad I_3 = \det\mathbf{C} = (\det\mathbf{F})^2. \tag{1}$$

The invariants I_4 and I_5 account for the existence of the fiber reinforcement. Let the unit vector $\mathbf{A}$ denote the fiber direction in the undeformed configuration. Then, I_4 and I_5 are defined by

$$I_4 = \mathbf{FA} \cdot \mathbf{FA} = \mathbf{F} \cdot (\mathbf{CA}), \quad I_5 = \mathbf{F} \cdot (\mathbf{C}^2\mathbf{A}). \tag{2}$$

Let $\mathbf{a} = \mathbf{FA}$, which is the image of $\mathbf{A}$ under the deformation (for a homogeneous deformation). It is clear from (2) that $\sqrt{I_4}$ is the stretch in the direction of the fiber reinforcement. The invariant I_4 therefore registers only deformations that modify the length of the fiber. For example, if the vector $\mathbf{A} = \mathbf{i}_1$, where $\mathbf{i}_1$ is a Cartesian coordinate axis, is the direction of reinforcement in the undeformed configuration, then $I_4 = C_{11}$, while $I_5 = C_{11}^2 + C_{12}^2 + C_{13}^2$. Thus, in this case the invariant I_5 registers changes in the fiber length through the indicator C_{11} and also shearing deformations through the indicators C_{12} and C_{13}.

For augmented isotropic nonlinearly elastic material models with unidirectional reinforcing the most general strain-energy function is given by

$$W = W_{\text{iso}}(I_1, I_2, I_3) + W_{\text{fib}}(I_4, I_5). \tag{3}$$

The first term in (3) represents the *isotropic base material*, while the second term is the so-called *reinforcing model*. The strain energy is normalized so that it is zero in the undeformed configuration and the stress vanishes there. These two conditions put restrictions on the forms of W_{fib} and W_{iso}. We restrict W_{fib} further to functions that depend on only one invariant, i.e. we consider reinforcing models of the forms $F(I_4)$ and $G(I_5)$ for appropriate choices of the functions F and G. Since the strain energy is normalized to zero in the undeformed configuration they satisfy $F(1) = 0$ and $G(1) = 0$. The assumption that the stress vanishes in the undeformed configuration yields the restrictions $F'(1) = 0$ and $G'(1) = 0$. Additional restrictions are imposed by taking the strain energy to be non-negative, and we assume that $F''(1) \geq 0$ and $G''(1) \geq 0$.

Our concern here is with the ellipticity of these material models under plane deformations. The fiber reinforcement is taken to lie in the plane of interest. We obtain conditions on $F(I_4)$ and $G(I_5)$ that provide a qualitative understanding of the ellipticity status of the model (3). For planar deformations in a fiber containing plane, the displacement field $\mathbf{u}$ is of the form $\mathbf{u} = \mathbf{x} - \mathbf{X} = u_1(X_1, X_2)\mathbf{i}_1 + u_2(X_1, X_2)\mathbf{i}_2$, so that $F_{13} = F_{23} = F_{31} = F_{32} = 0$, and $F_{33} = 1$. Therefore, the components of $\mathbf{C}$ satisfy $C_{13} = C_{23} = 0$, and $C_{33} = 1$. Henceforth, Greek indexes take the values 1 and 2. The governing partial differential equations of equilibrium for the displacement field $\mathbf{u}(\mathbf{X})$ in a homogeneous compressible elastic material can be written in the standard form

$$A_{\alpha\beta\gamma\delta} u_{\gamma,\delta\beta} = 0, \tag{4}$$

where

$$A_{\alpha\beta\gamma\delta} = \frac{\partial^2 W}{\partial F_{\alpha\beta} \partial F_{\gamma\delta}}. \tag{5}$$

The quasi-linear system of differential equations (4) is elliptic at the solution $\mathbf{u}$ and at a point $\mathbf{X}$ if and only if

$$\det \mathbf{Q}(\mathbf{n}) \neq 0, \tag{6}$$

for every unit vector $\mathbf{n} = (n_1, n_2)$, where

$$Q_{\alpha\gamma} = A_{\alpha\beta\gamma\delta}(\mathbf{F}(\mathbf{X}))n_\beta n_\delta, \tag{7}$$

$\mathbf{Q}$ being the acoustic tensor (which is symmetric).

Similarly, the equilibrium equations for the displacement field $\mathbf{u}(\mathbf{X})$ in a homogeneous incompressible elastic material can be written as

$$A_{\alpha\beta\gamma\delta}u_{\gamma,\delta\beta} - p_{,\beta} = 0, \tag{8}$$

where $p(\mathbf{X})$ is a scalar pressure field associated with the incompressibility constraint $\det \mathbf{F} = 1$. Now, the quasi-linear system of differential equations (8) is elliptic at the solution $\mathbf{u}$ and at a point $\mathbf{X}$ if and only if

$$\det \begin{bmatrix} \mathbf{H}(\mathbf{n}) & -\mathbf{n} \\ \mathbf{n}^T & 0 \end{bmatrix} \neq 0 \tag{9}$$

for every unit vector $\mathbf{n} = (n_1, n_2)$, where

$$H_{\alpha\beta} = F_{\gamma\alpha}Q_{\gamma\delta}F_{\delta\beta}. \tag{10}$$

Analysis of equation (6) (for a compressible material) and (9) (for an incompressible material) for a specific form of the strain-energy function W enables its ellipticity status to be determined for every unit vector $\mathbf{n}$. If (6) (or (9)) is satisfied the deformation in question is said to be elliptic for the considered strain energy. If, for a given strain energy, all admissible deformations are elliptic, then the material itself is said to be elliptic. On the other hand, for a given strain-energy function, if, for a specific deformation gradient $\mathbf{F}$, equation (6) (or (9)) is not satisfied for some unit vector $\mathbf{n}$, then the deformation is said to be non-elliptic for that material model. Any unit vector $\mathbf{n}$ for which ellipticity fails identifies the normal vector to a weak surface, across which one or more of the differentiability requirements required in the derivation of the strong form (4) (or (8)) of the field equations are not satisfied.

The Effect of I_4 or I_5 Reinforcement

Here we summarize briefly results for reinforcing models of the forms $W_{\text{fib}}(I_4, I_5) = F(I_4)$ and $W_{\text{fib}}(I_4, I_5) = G(I_5)$. In particular, if the reinforcement is strong then the terms in $F(I_4)$ and $G(I_5)$ are dominant in the respective strain energies and stresses. It therefore suffices to focus on these contributions in the analysis of the ellipticity and failure of ellipticity.

Beginning with $F(I_4)$, following [8], we take F to satisfy $F(1) = F'(1) = 0$ and $F''(1) \geq 0$, as already noted. Moreover, we assume that $F'(I_4) > 0\,(< 0)$ for $I_4 > 1\,(< 1)$, so that the contribution of F to the stress in the fiber direction is tensile (compressive) for extension (contraction) of the fiber length. Then, if the isotropic part $W_{\text{iso}}(I_1, I_2, I_3)$ (for a compressible material) or $W_{\text{iso}}(I_1, I_2)$ (for an incompressible material) of the strain energy is taken to be strongly elliptic, the inequalities

$$F'(I_4) \geq 0, \quad F'(I_4) + 2I_4F''(I_4) \geq 0 \tag{11}$$

are sufficient to guarantee overall strong ellipticity of the strain energy, whether the material is compressible or incompressible. Thus, necessary conditions for failure of ellipticity are

$$\text{either} \quad F'(I_4) < 0, \quad \text{or} \quad F'(I_4) + 2I_4F''(I_4) < 0. \tag{12}$$

Since $F'(I_4) < 0$ in compression it follows (for both compressible and incompressible materials) that fiber failure is to be expected in fiber compression. Under compressive loading in the fiber direction the associated incipient loss of ellipticity may be related to fiber kinking, the weak surface (of discontinuity) being (almost) normal to the fiber directions. In fiber extension we have $F'(I_4) > 0$ so that a necessary condition for loss of ellipticity in extension is that $F''(I_4) < 0$, which corresponds to loss of convexity of the reinforcing model. In this case the weak surface associated with loss of ellipticity is (close to) parallel to the fiber direction and we may identify the failure mode as fiber de-bonding.

The situation in respect of the reinforcing model $G(I_5)$ is somewhat different. While $G(I_5)$ is taken to satisfy conditions analogous to those given above for $F(I_4)$, the resulting strong ellipticity and loss of ellipticity conditions are more complicated than those for $F(I_4)$ given in (11) and (12) and are not described here. A detailed account, beyond the basic plane-strain analysis given in [8] and [9] for incompressible and compressible materials respectively, will be provided elsewhere. Note that in [9] the argument of G was taken to be a simplified modification of I_5 rather than I_5 itself.

The results show that loss of ellipticity is to be expected regardless of the particular form of the reinforcing model $G(I_5)$. Under compressive loading in the fiber direction, fiber kinking is again possible, as for $F(I_4)$, but, additionally, a weak surface parallel to the fiber direction is possible, in which case the associated failure mode may be interpreted as fiber splitting. Moreover, for certain specific deformations fiber kinking and splitting may be predicted to occur simultaneously. This is so for both compressible and incompressible materials. Under tensile loading in the fiber direction for compressible materials, weak surfaces parallel to the fiber direction (again associated with de-bonding) or normal to the fiber direction may arise, the latter being associated with a failure mode identified as matrix failure. These possibilities do not depend on G losing convexity. Such failure mechanisms, however, do not arise in tension in the incompressible case if G is convex. Failure for incompressible materials in tension is associated with weak surfaces that are neither parallel to nor perpendicular to the fiber direction.

Acknowledgement

This research was supported by the Spanish Ministerio de Ciencia y Tecnologia, PLAN NACIONAL, MAT:2002-03240.

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