Determination of crack initiation direction from a bi-material notch based on the strain energy density concept

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Summary

The article presents a procedure for the determination of the direction of crack initiation from a bi-material notch based on knowledge of the strain energy density distribution. First, the strain energy density for a bi-material notch is expressed and then the directions of the crack initiation are evaluated for the varying ratio of Young's moduli E_1/E_2 of both materials.

Introduction

In practical engineering structures, geometrical and material discontinuities are frequently responsible for their final failure. Most of such discontinuities can be mathematically modelled as bi-material notches (Fig. 1).

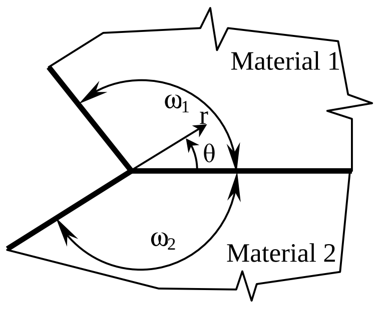

Fig. 1 Bi-material notch with corresponding polar coordinate system

In the following article a bi-material notch is analysed from the perspective of linear elastic fracture mechanics, i.e. the validity of small scale yielding conditions is assumed. It is further assumed that the bi-material interface is of welded type and the notch radius R $\rightarrow$ 0 (sharp bi-material notch).

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Stress distribution in the vicinity of a bi-material notch

The expressions for the singular stress distribution referring to plane problems in the vicinity of a bi-material notch are introduced in this chapter. The results are based on the solution of Airy stress function. The singular stress components can be written (in polar coordinates *r, q*, see Fig.1) in the following form:

$$s_{irr} = -\frac{H_k}{\sqrt{2p}} r^{l_k - 1} l_k (a_{ik} \sin((l_k+1)q) + b_{ik}\cos((l_k+1)q) - 3c_{ik}\sin((l_k-1)q) - 3d_{ik}\cos((l_k-1)q) +$$
$$+ l_k a_{ik}\sin((l_k+1)q) + l_k b_{ik}\cos((l_k+1)q) + l_k c_{ik}\sin((l_k-1)q) + l_k d_{ik}\cos((l_k-1)q))$$
$$s_{iqq} = \frac{H_k}{\sqrt{2p}} r^{l_k - 1}(l_k+1)(a_{ik}\sin((l_k+1)q) + b_{ik}\cos((l_k+1)q) + c_{ik}\sin((l_k-1)q) + d_{ik}\cos((l_k-1)q))\, l_k$$
$$s_{irq} = -\frac{H_k}{\sqrt{2p}} r^{l_k - 1} l_k (a_{ik}\cos((l_k+1)q) - b_{ik}\sin((l_k+1)q) - c_{ik}\cos((l_k-1)q) + d_{ik}\sin((l_k-1)q) +$$
$$+ l_k a_{ik}\cos((l_k+1)q) - l_k b_{ik}\sin((l_k+1)q) + l_k c_{ik}\cos((l_k-1)q) - l_k d_{ik}\sin((l_k-1)q)) \qquad (1)$$

where the values of l_k are in the interval (0; 1). The subscript *i* refers to material 1 or 2. The value H_k is the so called generalized stress intensity factor (GSIF) and its value results from a numerical solution for a certain construction with the notch and with given boundary conditions. The coefficients a_{ik}, b_{ik}, c_{ik}, d_{ik} for i = 1, 2 are known parameters corresponding to l_k and depending on the material combination and notch geometry. Generally, there exist one or two singularities of type (1) corresponding to one or two different values of l_k (k = 1 or k = 1, 2).

Strain energy density

The stress state (1) leads inherently to a combined mode of loading. In such cases, it is suitable to use strain energy density (SED) [1], [2] to describe crack behaviour. In 1991 Sih and Ho showed that damaging of a material should be estimated using strain energy density *dW/dV* independently of the power of notch singularity. Furthermore, they showed that fracture initiation is associated with a critical value *(dW/dV)c* which is a material characteristic.

Strain energy density *w* = *dW*/*dV* is defined by the equation:

$$dW / dV = \int_0^e s \, de \qquad (2)$$

where σ and ε represent stress and strain components, respectively. If we limit ourselves to plane problems and consider polar coordinates, the above equation becomes to:

$$(dW/dV)_i = w_i = 2\sigma_{i\theta\theta}\,\sigma_{irr}(\bar{\kappa}_i - 1) + (\sigma_{i\theta\theta}{}^2 + \sigma_{irr}{}^2)(\bar{\kappa}_i + 1) + 4\sigma_{ir\theta}{}^2 \Big/ 8\mu_i \qquad (3)$$

where $\bar{\kappa}_i = (1-\nu_i)/(1+\nu_i)$ for plane stress and $\bar{\kappa}_i = (1-2\nu_i)$ for plane strain; μ_i is shear modulus and ν_i is the Poisson ratio of the material *i*.

Distribution of the strain energy density

The expression for SED at the bi-material notch tip can be derived by substitution of the expressions for the stresses (1) into formula (3). Considering the existence of one singularity only (subscript k = 1 is omitted, i.e.: $H_k = H$, $\lambda_k = \lambda$, $a_{ik} = a_i$, etc.) we get

$$\begin{aligned} w_i = {} & \frac{1}{4} H^2 r^{(2\lambda-2)} \lambda^2 [2a_i \sin((\lambda+1)\theta)\left(c_i \sin((\lambda-1)\theta) + d_i \cos((\lambda-1)\theta)\right)\left(\lambda^2 - 1\right) + \\ & +2b_i \cos((\lambda+1)\theta)\left(c_i \sin((\lambda-1)\theta) + d_i \cos((\lambda-1)\theta)\right)\left(\lambda^2 - 1\right) + \\ & +2a_i \cos((\lambda+1)\theta)\left(c_i \cos((\lambda-1)\theta) - d_i \sin((\lambda-1)\theta)\right)\left(\lambda^2 - 1\right) + 2b_i \sin((\lambda+1)\theta) \\ & \left(d_i \sin((\lambda-1)\theta) - c_i \cos((\lambda-1)\theta)\right)\left(\lambda^2 - 1\right) + a_i^2(1+2\lambda+\lambda^2) + b_i^2(1+2\lambda+\lambda^2) + \\ & +c_i^2(1-2\lambda+\lambda^2+4\bar{\kappa}_i - 4\cos((\lambda-1)\theta)^2\bar{\kappa}_i) + d_i^2(1-2\lambda+\lambda^2 + \\ & +4\cos((\lambda-1)\theta)^2\bar{\kappa}_i) + 8c_i \sin((\lambda-1)\theta) d_i \cos((\lambda-1)\theta)\bar{\kappa}_i]/(\pi\mu_i) \end{aligned} \qquad (4)$$

where subscript i = 1 or 2 and it refers to the corresponding material. Note that if two singularities exist, the relation for SED can be obtained analogically, but – due to the complicated form – it is not presented here.

Determination of direction of initial crack propagation

Following the basic assumption of SED theory we suppose that the direction θ_m of crack initiation will be identical with the direction of the local minimum of strain energy density $w(r, \theta)$.

$$w(r,\theta_m) = \min(w_i(r,\theta)) \qquad (5)$$

and the value of crack propagation angle θ_m is then given by

$$\left(\frac{\partial w}{\partial \theta}\right)_{\theta_m} = 0\ , \qquad \left(\frac{\partial^2 w}{\partial \theta^2}\right)_{\theta_m} > 0\,. \qquad (6)$$

It can be easily shown that the direction of crack initiation is independent of the absolute value of GSIFs and depends only on their ratio H_2/H_1. The ratio results from the numerical solution of the body with a notch. Generally the value of angle q_m depends on the distance r where the conditions (6) are applied. The dimension r has to be chosen depending on the corresponding rupture mechanism and the microstructure of the material. It can be taken as a dimension of a plastic zone in the case of cyclic loading or as a grain size in the case of brittle fracture, etc.

Numerical example

The rectangular bi-material notch (see Fig. 2) was studied in detail. The direction of the minimum value of SED was evaluated for a varying ratio of Young's moduli $E_1/E_2 \in \langle 0.1;\ 11\rangle$ and on a varying ratio of generalized stress intensity factors $H_2/H_1 \in \langle 0.1;11\rangle$. The dimensional parameter r was taken as r = 0.001m, Poisson's ratio was constant at ν = 0.3, and the case of plane stress was considered.

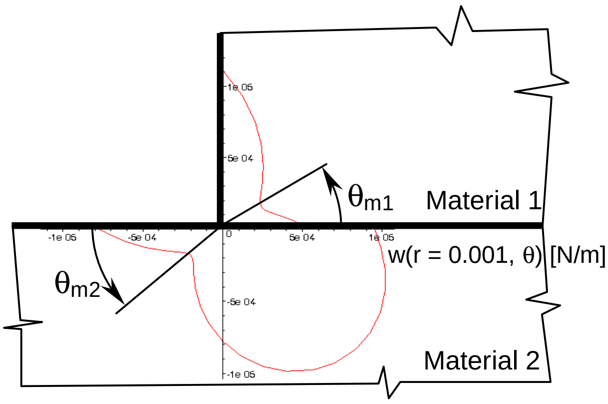

Fig. 2 Strain energy density distribution around the rectangular bi-material notch (ω_1 = 90°, ω_2 = 180°, H_2/H_1 = 5, E_1/E_2 = 4)

The results are shown in Fig. 3 and 4. It is seen that generally two possible directions of w_{min} occur. Usually one of them is in material 1 and the second belongs to material 2. Note that the crack will initiate only in the case where the value of w_{min} is greater than its critical value w_c that is a material constant.

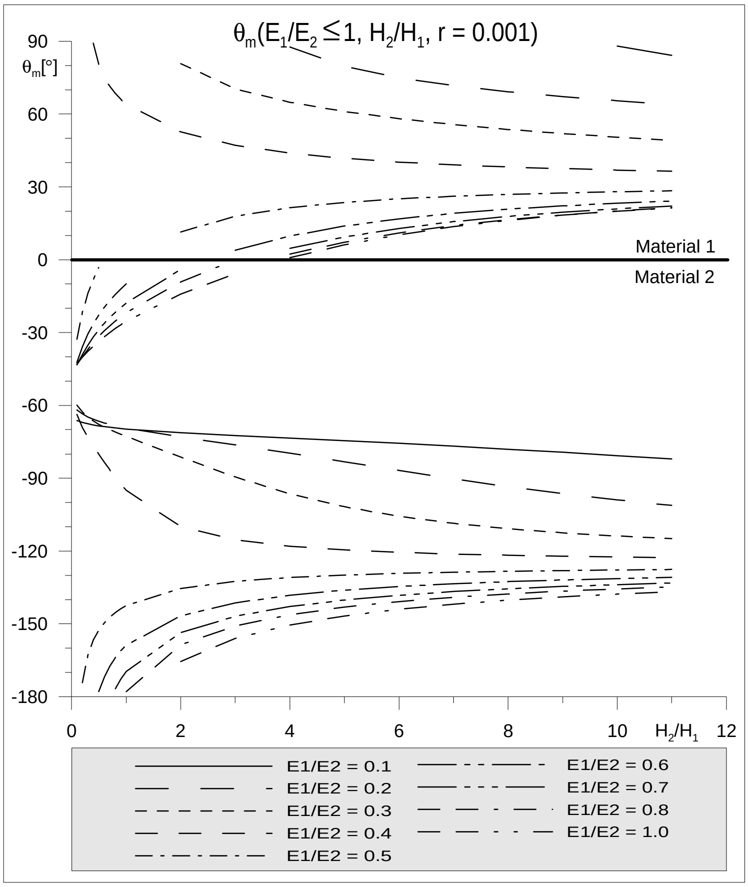

Fig. 3: Directions of crack initiations for $E_1/E_2 \in \langle 0.1; 1\rangle$

Conclusion

The procedure for the determination of the direction of crack initiation from a bi-material notch based on knowledge of the strain energy density distribution has been presented. The procedure makes it possible to assess the behaviour of a crack growing in composite materials and it can be used to increase the reliability of service-life estimation.

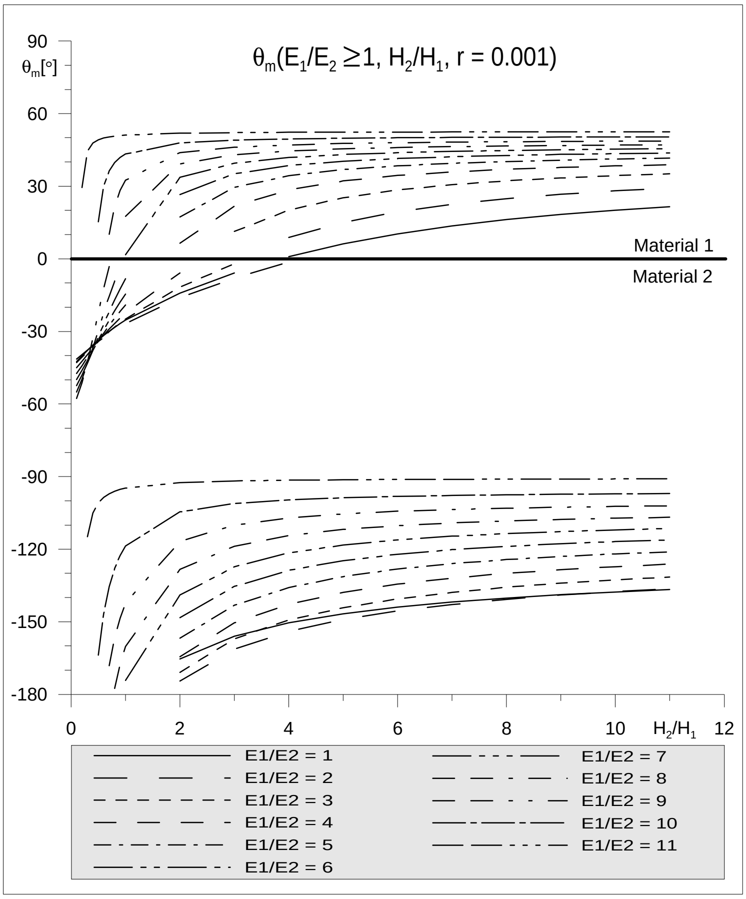

Fig. 4: Directions of crack initiations for $E_1/E_2 \in \langle 1; 11 \rangle$

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Reference

1. Sih G.C., Ho, J.W. (1991): “Sharp notch fracture strength characterized by critical energy density”, In: *Theoretical and Applied Fracture Mechanics*, 16: (3), pp. 179 – 214
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