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# Towards Hybrid Equilibrium Models for Large Displacements of Plates

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## Summary

This paper considers the use of hybrid equilibrium finite element models for the analysis of plate bending problems when small strains and linear material behaviour remain valid but the displacements become large enough to activate membrane forces. The linear relations for a quadrilateral hybrid element are reviewed, and comparison is made against a conforming displacement finite element to illustrate the power of hybrid models. The use of a recent co-rotational technique in large displacement analysis is then demonstrated with reference to the conforming element, and its extension to the hybrid element is briefly discussed. The ultimate objective is to realise solutions for large displacements which satisfy equilibrium in a strong point by point sense. Some examples are included to illustrate the concepts.

## Introduction

Rigid body displacements are the main quantities to be accounted for in modelling large displacement behaviour in the presence of small elastic strains. Equilibrium finite element models capable of producing strong local forms of equilibrium were initially developed by the Liege school [1]. Conceptual advances were made by Almeida *et al.* in the form of hybrid elements with discontinuous boundary displacements [2]. Problems associated with spurious kinematic modes in linear analysis have been dealt with by using special solvers, or by developing the macro-element concept [3]. This form of hybrid element enables for example a quadrilateral plate element to be formed with 4 corner nodes to define its shape, and 4 midside nodes with which to associate generalised displacement and force variables related by a stiffness matrix [4]. The way is now open to exploiting the co-rotational approach for modelling large displacements as exemplified by Izzuddin [5] which is largely independent of the specific element formulation.

The structure of the paper continues as follows: a brief review of the linear formulation of a hybrid plate element is followed by a numerical example to illustrate the nature of the displacements that are recovered in the case of small displacements. Then co-rotational concepts are introduced and the example is re-analysed for the case of large displacements but using fully conforming elements. Conclusions are drawn, and the further work required to implement non-linear analysis with the hybrid elements is described.

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**Linear formulation of a hybrid equilibrium plate element**

The essential ingredients of the hybrid element in the present work are internal stress fields which are statically admissible but generally semi-continuous, and boundary displacement fields which are defined independently for each side, and thus are generally discontinuous at the corners of an element. By relaxing the continuity conditions for the polynomial fields compared to the more conventional hybrid elements it is possible to maintain a strong form of equilibrium without the presence of spurious kinematic modes.

$$\{\sigma\} = [\boldsymbol{S}]\{\boldsymbol{s}\}, \text{ and } \{\delta\} = [\boldsymbol{V}]\{\boldsymbol{v}\} \tag{1}$$

$\sigma$ and $\delta$ represent polynomial fields of stress-resultants and boundary displacements with parameters $\boldsymbol{s}$ and $\boldsymbol{v}$ respectively. The parameters $\boldsymbol{v}$ represent modes of side displacement for all sides of an element, and the modes are based on complete Legendre polynomials up to degree $p$ which is the same as the degree of the stress-resultants. Kinematic degrees of freedom are quantified by $\{\boldsymbol{v}\}$, and are associated with midside nodes of an element. The modes of side displacement are selected to correspond to the rigid body modes and deformation modes.

Dual modes of side traction $\boldsymbol{g}$ correspond to resultant forces and/or moments and self-balancing modes. General distributions of tractions $\boldsymbol{t}$ are represented by the modes defined in Equation (2). Particular distributions of traction which equilibrate with $\sigma$ are denoted by $\bar{\boldsymbol{t}}$, and these are related to the stress parameters as in Equation (2).

$$\{\boldsymbol{g}\} = \oint_{\Gamma} [\boldsymbol{V}]^T \{\boldsymbol{t}\} d\Gamma, \text{ and } \{\bar{\boldsymbol{t}}\} = [\bar{\boldsymbol{S}}]\{\boldsymbol{s}\} \tag{2}$$

Weak compatibility conditions between internal strains and side displacements are expressed by (body forces and initial strains are omitted for simplicity):

$$\int_{\Omega} [\boldsymbol{S}]^T \{\varepsilon\} d\Omega = \oint_{\Gamma} [\bar{\boldsymbol{S}}]^T \{\delta\} d\Gamma \quad \text{or } [\boldsymbol{F}]\{\boldsymbol{s}\} = [\boldsymbol{D}]^T \{\boldsymbol{v}\} \tag{3}$$

where $\{\varepsilon\} = [\boldsymbol{f}]\{\sigma\}, [\boldsymbol{F}] = \int_{\Omega} [\boldsymbol{S}]^T [\boldsymbol{f}][\boldsymbol{S}] d\Omega$, and $[\boldsymbol{D}]^T = \oint_{\Gamma} [\bar{\boldsymbol{S}}]^T [\boldsymbol{V}] d\Gamma$.

When tractions $\boldsymbol{t}$ are polynomials of degree $\leq p$, strong equilibrium between side tractions and internal stress fields is enforced by:

$$\oint_{\Gamma} [\boldsymbol{V}]^T \{\bar{\boldsymbol{t}}\} d\Gamma = \oint_{\Gamma} [\boldsymbol{V}]^T \{\boldsymbol{t}\} d\Gamma \text{, i.e. } [\boldsymbol{D}]\{\boldsymbol{s}\} = \{\boldsymbol{g}\} \tag{4}$$

The stiffness matrix [$\boldsymbol{K}$] of an element is formed by elimination of $\{\boldsymbol{s}\}$ so that:

$$[\boldsymbol{K}]\{\boldsymbol{v}\} = \{\boldsymbol{g}\} \text{ where } [\boldsymbol{K}] = [\boldsymbol{D}][\boldsymbol{F}]^{-1}[\boldsymbol{D}]^T \tag{5}$$

**Example of a plate problem**

A 15m by 25m rectangular plate with thickness 0.333m is considered. The edges of length 15m are clamped, but may be unrestrained against pull-in, and the edges of length 25m are unsupported. A uniformly distributed load is applied over a 3m wide strip along one of the unsupported edges. Reissner-Mindlin theory is assumed with elastic material properties, $E$ = 30kN/mm$^2$, and $\nu$ = 0.2.

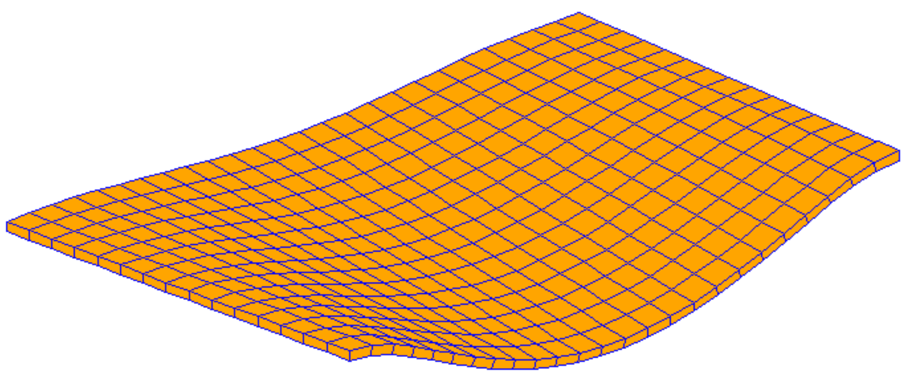


Figure 1: Deflected form of the plate when large displacements are developed without restraint against pull-in.

The plate problem was modelled by both hybrid equilibrium and conforming elements for the linear phase of behaviour for a load of 10kPa. The hybrid models are used in two forms: (a) self-balancing traction modes are released and only the rigid body kinematic freedoms are included in compatibility equations; and (b) all quadratic modes of rotation and linear modes of transverse displacement are included. The conforming model is based on a 4-node element with bilinear shape functions and additional quadratic hierarchic freedoms for the transverse displacement field. Comparisons of deflections and strain energies in Table 1 indicate that the hybrid models can provide good quality solutions, which give rise to the expectation that the hybrid side displacements can effectively control the large displacement problem in a Newton-Raphson type of analysis.

Table 1: Deflections and strain energies for the linear models.

| Model | Maximum deflection mm | Strain energy kNm |
|---|---|---|
| 5×9 conforming | 43.65 | 7.635 |
| 15×25 conforming | 47.55 | 8.466 |
| 15×25 hybrid(b) | 48.06 | 8.576 |
| 15×25 hybrid(a) | 48.31 | 8.616 |
| 5×9 hybrid (b) | 48.10 | 8.624 |
| 5×9 hybrid(a) | 49.13 | 8.799 |

## Co-rotational formulation for large displacement analysis

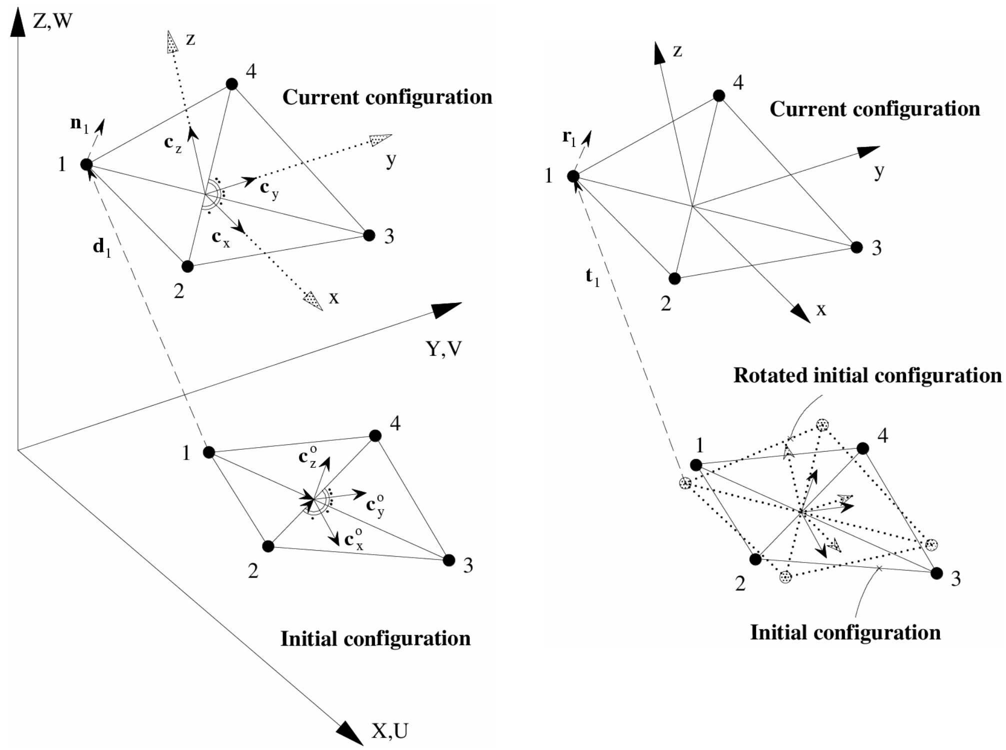
The co-rotational technique for large displacement analysis eliminates rigid body modes, particularly rigid body rotations, from the global element deformations, resulting in local modes for which linear analysis principles may be employed. Through geometric transformations between the local and global systems, the linear local element response is automatically upgraded to model geometric nonlinearity, such transformations being applicable to a wide range of elements with similar degrees of freedom.

A new co-rotational approach was recently proposed by Izzuddin [5] for the conforming 4-node plate element. A main advantage of this co-rotational approach lies in the choice of the local reference system, where the local x and y axes bisect the element diagonals in the deformed configuration, as depicted in Figure 2. This is shown to have the beneficial effect of leading to a symmetric global tangent stiffness matrix [5]. In addition, the conforming element employed only two rotational degrees of freedom per node, conveniently defined as the two smallest components of the initial normal to the mid-plane.

The above co-rotational approach can be readily modified to incorporate the hybrid equilibrium element discussed previously, though two main issues will requires careful consideration. Firstly, the global displacement parameters of the hybrid element are based on midside nodes, and hence the local x and y axes would need to be re-defined as the bisectors of the two lines joining the two sets of opposite nodes. Secondly, the hybrid element utilises three rigid body rotational freedoms per side, and therefore the full set of rotational freedoms is required to take into account the effect of large rotations.

## Example of the plate problem extended to large displacements

The same example is now considered for large displacements based on the 15×25 conforming model using the current ADAPTIC [5] software. Due to the flexibility of this software, based on the co-rotational formulation, to embrace general forms of element, it is intended to include the hybrid equilibrium element.




(a) Global freedoms (b) Local freedoms

Figure 2: Global and local co-rotational reference system

For these analyses the plate is considered with the supported edges restrained and unrestrained against horizontal pull-in. The development of the non-linear relations between the maximum vertical deflection, at the mid-span position of the loaded edge, and the load intensity are shown in Figure 3. Also shown for comparison is the extension of the linear relation which is unaffected by the restraint condition. Figure 1 also illustrates the unrestrained case under a load of 1000kPa.

## Conclusions

For linear behaviour, the hybrid equilibrium plate element can provide reliable solutions for side displacements, although such displacements are not strictly conforming.

The linear hybrid element, with a stiffness matrix referring to midside nodes, is suitable for a co-rotational approach to the analysis of geometric non-linear behaviour.

It is planned to incorporate the hybrid plate element in the ADAPTIC software after appropriate modifications to account for redefinitions of local axes and rotational freedoms.

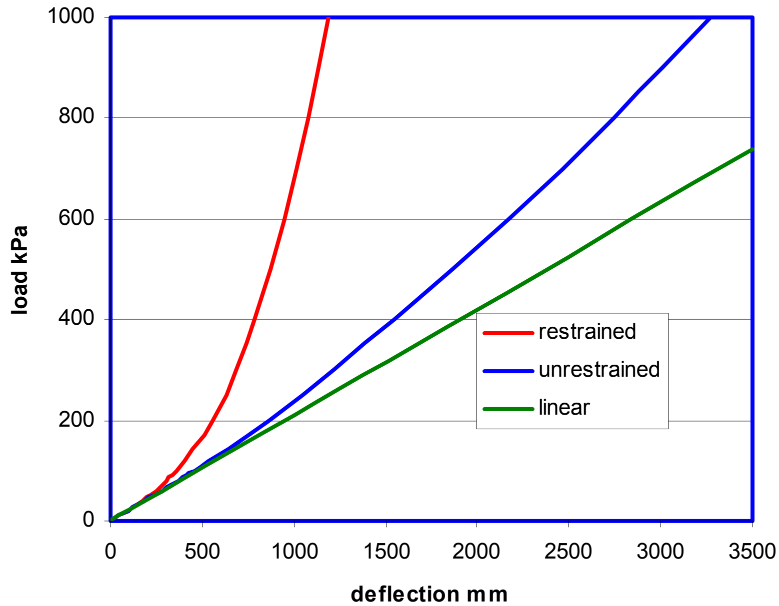



Figure 3: Equilibrium paths for linear and non-linear deflections.

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