Multi-Scale Linear Stability Analysis

B. C. Houchens[1] and J. S. Walker[1]

Summary

We have developed a technique for investigating the linear stability of flows which are difficult or impossible to tackle with numerical analysis alone. This method utilizes three solutions. Complementing the standard numerical solution is an asymptotic solution in some parameter extreme, and a hybrid solution which combines numerical and analytical techniques. The composite of these three solutions gives a stability analysis which covers a complete parameter range. The results of this method are presented here for two problems: the confined Rayleigh-Bénard flow, and the float-zone liquid bridge model, both in cylindrical coordinates with electrically conducting incompressible fluids. In both cases, a uniform, axial magnetic field is applied, and is the parameter which is varied. It is increased from zero to infinite strength, and the critical driving force required for neutral stability is found.

Introduction

Linear stability analysis offers the most insight when used to investigate flow transitions over large parameter spaces. However, many numerical issues arise, particularly when dealing with a large number of independent variables. This is the case in liquid metal magnetohydrodynamics, where the hydrothermal and electro-magnetic problems are coupled, forcing a simultaneous solution in nine primitive variables. When the base flow can be represented analytically, as in the Rayleigh-Bénard problem, the perturbation can still be numerically challenging. This is especially true when magnetic field strengths are large, because this often confines the perturbation flow to thin regions, requiring high resolution. For problems such as the float-zone model, the base flow must be solved numerically. In this case, even obtaining accurate base flows can involve impractical computational costs. Small errors in the base flow can quickly deteriorate into nonphysical predictions from the stability analysis. Numerical breakdown typically occurs when thin thermal or viscous boundary layers are present in the base flow. To overcome these numerical obstacles, we have developed a multi-scale linear stability method to treat flows which are segregated into thin boundary layers, and large regions with small gradients.

The method requires a full numerical analysis at one end of the parameter range, an asymptotic analysis at the other extreme, and a hybrid of the two in the middle. In both problems, the fully numerical stability analysis is only feasible for zero and weak magnetic fields, because the magnetic field has a strong stabilizing effect. This stabilization occurs through the electromagnetic (EM) body force. When the magnetic field is switched on, it induces electric currents in the conducting fluid. These currents interact with the magnetic field to produce EM body forces in the flow.

[1] University of Illinois, Urbana-Champaign, Urbana IL 61801, USA

The body forces resist any fluid motion normal to the magnetic field (radial and azimuthal directions), thereby stabilizing the flow. The strength of the magnetic field is proportional to the dimensionless Hartmann number, Ha. As the magnetic field strength increases, more energy must be supplied to the flow to trip the instability.

We present results for two cases, both in cylindrical geometry and with Ha varying from zero to infinity. For both problems, the height of the liquid region is the same as the diameter. First, the linear stability of the finite Rayleigh-Bénard problem is investigated. In this case, the base flow is stagnant, and the base temperature profile is a linear conduction solution, independent of Ha. The stability of this problem with a uniform, vertical magnetic field was first studied by Touihri *et al.* [1] for Ha varying from 0 to 15. Here we extend the fully numerical analysis to Ha = 45. The stability of the infinite magnetic field strength Rayleigh-Bénard problem was treated by Chandrasekhar [2] for a liquid layer between two infinite horizontal isothermal walls. In the confined cylindrical geometry, we find an instability mechanism that is sub-critical to that observed by Chandrasekhar. Finally, we treat the range of Ha between 25 and 500 with the hybrid solution. With the three solutions, we are able to treat the confined Rayleigh-Bénard stability analysis for Ha from zero to infinity. Details of the instabilities are available in Houchens *et al.* [3].

The second case is the float-zone liquid bridge, neglecting buoyancy. This is a model of the float-zone process for semi-conductor crystal growth in micro-gravity. A parabolic heat flux is applied to the free surface, and the resulting temperature gradients cause surface tension variations. These drive thermocapillary (Marangoni) flow in the form of two axisymmetric, toroidal cells. If the temperature gradients on the free surface are too large, the thermocapillary driving force becomes too strong, and the axisymmetry is lost to three-dimensional disturbances [4]. The parameter which measures the strength of the driving force at the onset of this transition is the critical thermocapillary Reynolds number, Re_{cr}. Several experimental studies of the float-zone with and without magnetic fields have been undertaken, and these were recently reviewed by Dold *et al.* [5]. Kasperski *et al.* [6] studied the stability of the float-zone to axisymmetric disturbances, and Houchens and Walker [7] investigated non-axisymmetric transitions. However, linear stability analyses have yet to include magnetic damping effects. Here we present results of the fully numerical stability analysis for Ha from 0 to 100. We also discuss the asymptotic analysis, from which we find $Re_{cr} = O(\mathrm{Ha})$ for $\mathrm{Ha} \gg 1$. The hybrid solution is currently being formulated.

Problem Statements

We nondimensionalize the fluid velocity **v** using a viscous characteristic velocity, $V = \mu/\rho R$, where μ is the absolute viscosity, ρ is the density, and R is the dimensional radius. The scales for length, time, and pressure are R, R^2/ν, and $\rho\nu^2/R^2$ respectively, where ν is the kinematic viscosity defined as $\nu = \mu/\rho$. The electric potential ϕ is normalized by νB, where B is the magnitude of the applied magnetic flux density.

The electric current density **j** is normalized by σνB/R, where σ is the electrical conductivity. The temperature T is normalized by subtracting some reference temperature, and dividing by a characteristic temperature difference (ΔT). For the Rayleigh-Bénard problem, (ΔT) is the temperature difference between the top and bottom walls. For the float-zone, (ΔT) = qR/k, where q is the maximum value of the heat flux and k is the thermal conductivity. The magnetic field is applied in the axial direction, and we invoke the small magnetic Reynolds number approximation so that induced magnetic fields are negligible. With the Boussinesq approximation and neglecting viscous dissipation, the dimensionless governing equations are

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \bullet \nabla)\mathbf{v} = -\nabla P + \frac{Ra}{Pr} T\mathbf{e_z} + Ha^2(\mathbf{j} \times \mathbf{e_z}) + \nabla^2 \mathbf{v}, \qquad \nabla \bullet \mathbf{v} = 0, \qquad \text{(1a - d)}$$

$$\mathbf{j} = -\nabla\phi + \mathbf{v} \times \mathbf{e_z}, \qquad \nabla \bullet \mathbf{j} = 0, \qquad \frac{\partial T}{\partial t} + (\mathbf{v} \bullet \nabla)T = \frac{1}{Pr}\nabla^2 T \qquad \text{(1e - i)}$$

where $\mathbf{e_z}$ is the unit vector in the z-direction. The dimensionless parameters are the Rayleigh number, $Ra = g\beta(\Delta T)R^3/\nu\kappa$, the Prandtl number, $Pr = \nu/\kappa$, and the Hartmann number, $Ha = BR(\sigma/\mu)^{1/2}$, where $g = 9.81 m/s^2$, β is the volumetric expansion coefficient, and κ is the thermal diffusivity. In the float-zone problem, Ra = 0 and Pr = 0.02. The dimensionless boundary conditions for the Rayleigh-Bénard case are

$$\mathbf{v} = j_r = \frac{\partial T}{\partial r} = 0 \quad \text{at} \quad r = 1, \quad \text{and} \quad \mathbf{v} = j_z = 0, \quad T = \mp\frac{1}{2} \quad \text{at} \quad z = \pm 1. \qquad \text{(2a - j)}$$

For the float-zone model, the dimensionless boundary conditions are

$$v_r = j_r = 0, \quad \frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right) = -Re\frac{\partial T}{\partial \theta}, \quad \frac{\partial v_z}{\partial r} = -Re\frac{\partial T}{\partial z}, \quad \frac{\partial T}{\partial r} = 1 - z^2 \quad \text{at} \quad r = 1 \quad \text{(3a - e)}$$

$$\mathbf{v} = j_z = T = 0 \quad \text{at} \quad z = \pm 1 \qquad \text{(3f - j)}$$

where $Re = \rho R|d\gamma/dT|(\Delta T)/\mu^2$ is the thermocapillary Reynolds number, and |dγ/dT| is the magnitude of the decrease in surface tension with temperature.

Solution Technique and Results

We first apply a fully numerical linear stability analysis, with which we treat the range from no magnetic field (Ha = 0) to Ha = $O(10^2)$. We use spectral representations of both the base flow (for the float-zone) and perturbation variables, with the Chebyshev polynomials taken as the basis functions, and applied on a Gauss-Lobatto collocation grid. This grid places most of the collocation points in the region of largest gradients near r = z = 1. In both cases, the base flow exhibits axisymmetry, as well as symmetry about the mid-plane. Therefore, we need only treat the quadrant of the domain with $r \geqslant 0$ and $z \geqslant 0$. The axial symmetry allows us to break the stability analysis into two distinct problems: one with the same symmetry as the base flow, and one with the opposite. Doing so allows us to include only the even or odd polynomials in the axial representation of each variable. Compared to treating the full axial extent, we are able to halve the axial domain (and hence the grid), and double the number of nonzero coefficients in the representations. The final caveat in our numerical solution is our treatment of $r = 0$. The form of each variable can be found as r approaches zero. For example, the radial perturbation velocity v_{r1} goes as $r^{(m-1)}$, where m is the azimuthal wave number. We include this factor in the Chebyshev polynomial representation of each variable, and therefore use only the even Chebyshev polynomials in the radial direction. When the representations are substituted into the governing equations, at first order many of the equations reduce to zero at $r = 0$. However, because we have ensured the proper Taylor series expansion in r, we can simply apply the first non-zero order of the equations at $r = 0$. The increase in the grid and computational cost is negligible, but there is a critical improvement in the accuracy and smoothness of the representations. This modification, and the use of axial symmetry, allowed us to extend the fully numerical analysis of the float-zone from Ha = 20 (Re_{cr} = 5673) to Ha = 100 (Re_{cr} = 61,700).

At the other extreme of infinite magnetic field strength ($Ha \gg 1$), we use an asymptotic analysis. In this limit, the strong magnetic field confines the flow to distinct regions. Hartmann layers of thickness $O(Ha^{-1})$ exist along the top and bottom walls, and a parallel layer of thickness $O(Ha^{-1/2})$ occurs at $r = 1$. This parallel layer contains the physics necessary for the critical instability. In the $Ha \gg 1$ limit, the parallel layer is infinitely thin, and all curvature effects drop out. The central region of the domain is the $O(1)$ core. For both problems, the base flows in all regions can be solved analytically. It is also possible to deduce to what power of Ha the critical driving force must be raised to incite instability. In the Rayleigh-Bénard problem, $Ra_{cr} \sim Ha^{3/2}$. For the float-zone model, $Re_{cr} \sim Ha$. Once these relationships are known, the specific constants of proportionality can be found numerically.

At even moderate field strengths ($Ha \sim 100$), the magnetic field confines the perturbation flow out toward $r = 1$. Hartmann layers begin to form and axial gradients near $z = 1$ become difficult to resolve. The hybrid solution replaces the Hartmann layers with analytical expressions, but treats the rest of the domain numerically. In doing so, the curvature effects associated with a thicker parallel layer (at finite Ha) are included, while constraints at $z = 1$ are relaxed. The Hartmann layers contain all of the significant viscous diffusion of momentum in the axial direction. At $z = 1$, the no slip conditions are dropped, and the Hartmann conditions are applied on v_z and j_z. This implies that the axial viscous terms $\partial^2\mathbf{v}/\partial z^2$ must be dropped. This solution requires significant radial resolution, but allows the stability analysis to be extended beyond $Ha = O(10^2)$. The hybrid solution is justified by ensuring that there are regions of overlap with both the fully numerical and asymptotic solutions.

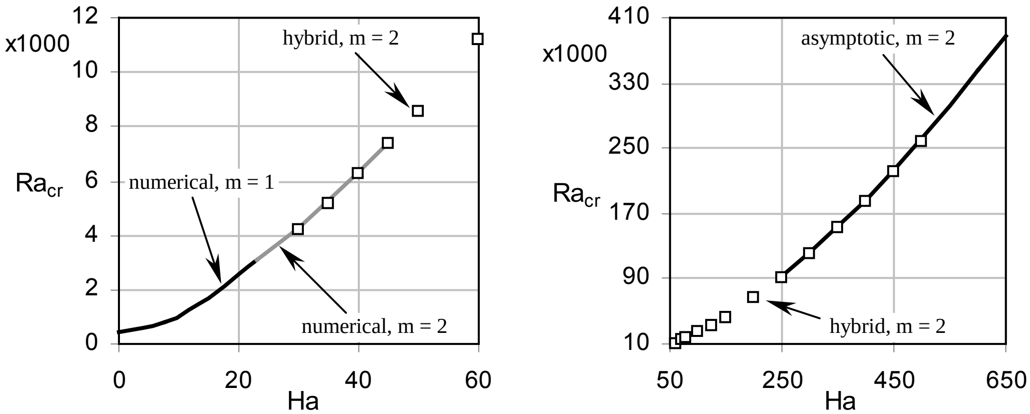

Figure 1: Ra_{cr} vs. Ha for the Rayleigh-Bénard problem. The azimuthal wave number is m.

For the Rayleigh-Bénard problem, all three solutions are presented in the Ra_{cr} versus Ha curve of figure 1. In the overlap region of the numerical and hybrid solutions, the difference in the values of Ra_{cr} decrease from 2.6% at Ha = 25 to 1.1% at Ha = 45. For the range of overlap between the hybrid and asymptotic solutions ($250 \leqslant \text{Ha} \leqslant 500$), the difference in the values of Ra_{cr} is less than 1.4%. In the infinite Rayleigh-Bénard problem, Chandrasekhar found that Ra_{cr} goes as Ha^2 in the limit of large Ha. In the finite case an instability with $Ra_{cr} = O(Ha^{3/2})$ occurs for $\text{Ha} \gg 1$. Because the side walls are electrically insulating, the radial electric current must go to zero at the walls. Therefore, the EM body force which resists azimuthal flow is very weak near the side walls. In the infinite case, there is no such weakening of the EM body force, hence the flow remains stabile to larger values of Ra. For the confined case, with $\text{Ha} \gg 1$, $Ra_{cr} = 23.48183Ha^{3/2}$. The azimuthal wave number m remains constant at 2 for $22.9 \leqslant \text{Ha} \leqslant \infty$. The core plays an important role in the heat transfer balance, and thus sets the scale of the instability mechanism. The size of the core changes little over the range $0 \leqslant \text{Ha} \leqslant \infty$, so m also changes very little.

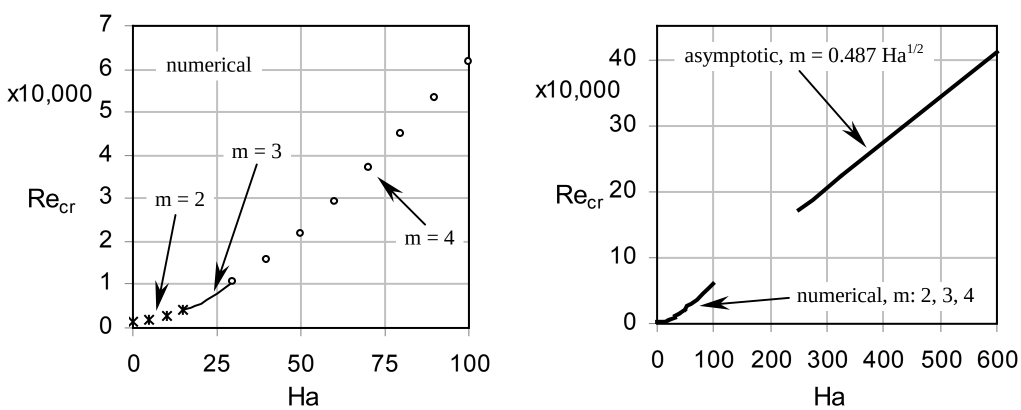

Figure 2: Re_{cr} vs. Ha for the float-zone problem. The azimuthal wave number is m.

The numerical and asymptotic critical curves for the float-zone problem are presented in figure 2. In this case, the parallel layer controls the instability, and the core is passive. As the Ha is increased, the parallel layer thins, and there are progressive shifts in m so that the instability circulation in the azimuthal direction remains on the order of the thickness of the parallel layer. This trend is also observed in the asymptotic solution, where Re_{cr} = 685Ha and m = 0.487$Ha^{1/2}$.

Conclusions

The multi-scale linear stability analysis has proved useful for treating flows where a large parameter space must be investigated. Full numerical and asymptotic solutions give good insight over confined parameter spaces. Often though, it is the intermediate range that is of most interest. Here, the hybrid solution is necessary. By treating boundary layers with analytical expressions, and the remaining domain numerically, it offers a bridge to the numerical and analytical solutions, not before available. This work was supported by NASA grants NAG 8-1453 and NAG 8-1705 and the National Computational Sciences Alliance under grant DMS030003N.

References

1 Touihri, R., Ben Hadid, H. and Henry, D. (1999): "On the Onset of Convective Instabilities in Cylindrical Cavities Heated from Below. II. Effect of a Magnetic Field", *Phys. Fluids*, **11**(8), 2089-2100.

2 Chandrasekhar, S. (1961): *Hydrodynamic and Hydromagnetic Stability,* Dover.

3 Houchens, B. C., Martin-Witkowski, L. and Walker, J. S. (2002): "Rayleigh-Bénard Instability in a Vertical Cylinder with a Vertical Magnetic Field", *J. Fluid Mech.*, **469**, 189-207.

4 Bohm, J., Lüdge, A., and Schröder, W. (1994): "Crystal Growth by Floating Zone Melting" in *Handbook of Crystal Growth, Vol. 2a: Bulk Crystal Growth, Basic Techniques*, D. T. J. Hurle, ed., Elsevier.

5 Dold, P., Schweizer, M., Cröll, A., and Benz, K. W. (2002): "Measurement of Microscopic Growth Rates in Float Zone Silicon Crystals", *J. Cryst. Growth*, **237-239**, 1671-1677.

6 Kasperski, G., Batoul, A., and Labrosse, G. (2000): "Up to the Unsteadiness of Axisymmetric Thermocapillary Flows in a Laterally Heated Liquid Bridge", *Phys. Fluids*, **12**(1), 103-119.

7 Houchens, B. C. and Walker, J. S. (2004) "Comparison of the Half-zone and Laterally Heated Full-zone Instability Mechanisms", under review for publication in the *J. Thermophysics and Heat Trans.*