# Non-axisymmetric Instabilities in Wide-Gap Spherical Couette Flow

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## Summary

We consider spherical Couette flow for aspect ratios $0.3 \le \beta \le 3$, and numerically compute the non-axisymmetric instabilities that arise. We show that they have the opposite equatorial symmetry as the basic state, and exhibit a progression from azimuthal wavenumber $m = 6$ at $\beta = 0.3$ to $m = 2$ at $\beta = 3$. We next consider the equilibration of these modes in the supercritical regime, and show that secondary bifurcations occur, beyond which one obtains solutions containing all wavenumbers, not just multiples of $m_0$. Finally, we include magnetohydrodynamic effects, and show that a surprisingly weak magnetic field may already have significant effects, such as switching the equatorial symmetry of the most unstable modes.

## Introduction

Spherical Couette flow is the flow induced in a spherical shell by differentially rotating the inner and/or outer spheres. Here we will consider only the simplest possible SCF scenario, namely where the outer sphere is fixed, and only the inner one rotates. The flow is then determined by two parameters, the aspect ratio $\beta = (r_o - r_i)/r_i$ describing the geometry, and the Reynolds number $Re = \Omega r_i (r_o - r_i)/\nu$ measuring the rotation rate.

For small aspect ratios, up to $\beta \approx 0.25$, the first instability (as $Re$ is slowly increased) is in the form of axisymmetric Taylor vortices. See for example [1] for some of these results. For larger aspect ratios one can still obtain Taylor vortices, but only if $Re$ is increased abruptly, or other special initial conditions are used [2,3]. If $\beta > 0.45$ though, no Taylor vortices are possible regardless of how the sphere is spun up.

It is thus of interest to consider what other instabilities might arise in this case. A number of experiments have been done [4,5,6], which reveal the first instability in this case to consist of a non-axisymmetric spiral wave, with wavenumbers in the range 3 to 6, depending on the aspect ratio. The linear onset of these instabilities has previously been computed [7,8] at particular aspect ratios, with good agreement with the experimental results. However, neither of these studies (nor any other work we are aware of) has considered the nonlinear equilibration of these instabilities in the supercritical regime. Such a study would be of considerable interest though, as some of the experiments [6] indicate that secondary mode transitions may occur in the sufficiently supercritical regime. The main purpose of this work, therefore, is to explore this supercritical regime, and discover whether any such secondary bifurcations can be obtained numerically as well.

## Linear Onset

Figure 1 shows $Re_c$ as a function of $\beta$, for the modes $m = 2$ to 6. See also [9]. We note how the most unstable mode decreases with increasing aspect ratio. The other point to note

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*Proceedings of the 2004 International Conference on Computational & Experimental Engineering & Science*
*26-29 July, 2004, Madeira, Portugal*

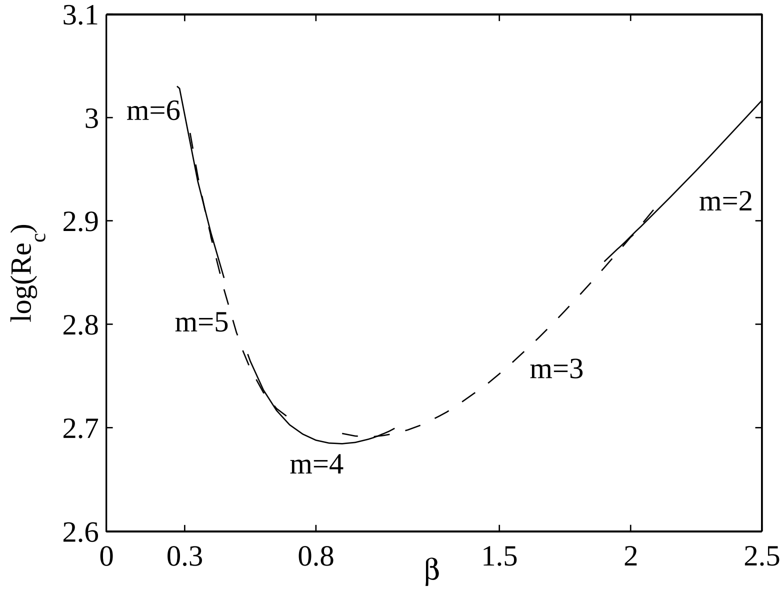


Figure 1: The logarithm of $Re_c$ as a function of β.

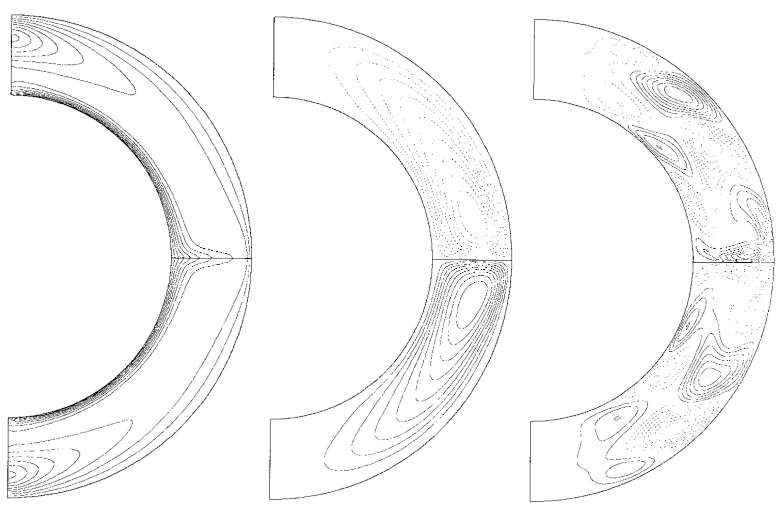

Figure 2: From left to right, contour plots of the angular velocity and meridional circulation of the axisymmetric basic state, and the ϕ-component of the non-axisymmetric instability, at $Re_c$, and for $\beta = 0.5$.

is that the equatorial symmetry of these modes is always the opposite of the basic state's. That is, where the axisymmetric basic state has $(U_r, U_\theta, U_\phi)$ symmetric, antisymmetric, symmetric about the equator, the non-axisymmetric instabilities have the opposite. Figure 2 shows an example of the basic state, and also the instability.

### Nonlinear Equilibration

The most thorough set of experiments [6] were done at $\beta = 1/3$ and 1/2, where according to Fig. 1 the wavenumbers are 6 and 5, respectively. These were also the values observed in the experiments, and indeed at the same $Re_c$ (to within 1–2%). In the supercritical regime the experiments then obtained mode transitions, in which the mode number is reduced by one. It would thus be of considerable interest to attempt to reproduce this behaviour numerically. Unfortunately, these particular aspect ratios turned out to be numerically inaccessible; the relatively thin gap width means very high resolution is required in $\theta$, and the relatively high $m_0$ means very high resolution is also required in $\phi$.

We therefore considered the numerically easier aspect ratios $\beta = 0.8$, 1.5 and 2.5, namely in the middle of the $m_0 = 4$, 3 and 2 ranges, and in each case pushed $Re$ as high as possible (between 1.5 and 2 times supercritical). In all three cases it was found that secondary bifurcations do indeed occur. However, rather than resulting in transitions to the next lower wavenumber, these bifurcations introduced structure in all modes, but with the dominant mode remaining the original $m_0$. It is possible of course that a true mode transition would still occur at even higher supercriticality. We suggest therefore that experiments should be done at some of these aspect ratios, to look for some of these bifurcations, and see whether mode transitions do eventually occur.

### Magnetic Couette Flow

If the fluid is electrically conducting, and one imposes a magnetic field (which for simplicity we will take to be purely vertical), one can obtain radically different solutions [10,11]. The equations to be solved in this case are

$$\frac{\partial \mathbf{U}}{\partial t} + Re\, \mathbf{U} \cdot \nabla \mathbf{U} = -\nabla p + \nabla^2 \mathbf{U} + M^2 (\nabla \times \mathbf{b}) \times \hat{\mathbf{e}}_z, \tag{1}$$

$$\nabla^2 \mathbf{b} = -\nabla \times (\mathbf{U} \times \hat{\mathbf{e}}_z), \tag{2}$$

where the Hartmann number $M$ measures the strength of the imposed field, and $\mathbf{b}$ is the induced field. What makes this problem particularly interesting is that it is known [11] that for $M = O(100)$ the non-axisymmetric instabilities have the same equatorial symmetry as the basic state, rather than the opposite. Somewhere between $M = 0$ and $O(100)$ the preferred instabilities must therefore switch from one symmetry to the other. We show that this transition occurs in the range $M \approx 10$, and again suggest further numerical and experimental work that could be done on this magnetic problem.

**Reference**

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