Bifurcation of vortex breakdown patterns

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Summary

Based on methods from bifurcation theory, we propose an efficient tool to analyze numerically obtained streamline patterns in steady vortex breakdown in a closed cylinder with two rotating covers. We show that the streamline patterns for small ratios of the angular velocities of the lid are organized around two codimension three degeneracies.

Introduction

Vortex breakdown is the creation of a secondary flow structure on a vortex axis. Vortex breakdown occurs in a number of important situations such as wingtip vortices and swirl burners. A very useful set-up for experimental and computational studies of vortex breakdown is a cylindrical container filled with fluid where one or both of the covers are rotating [4, 5]. The rotating covers create a main vortex along the cylinder axis, which may exhibit one or more vortex breakdowns of bubble type. In a large range of parameters, the flow is axisymmetric and hence it suffices to consider the intersection of the flow field with a meridional plane. Examples are shown in fig. 1. The parameters characterizing the problem are

$$Re = \frac{\Omega_1 R^2}{\nu},\ h = \frac{H}{R},\ \gamma = \frac{\Omega_2}{\Omega_1}, \tag{1}$$

where Ω_1, Ω_2 are the angular velocities of the bottom and the top cover, R, H are the radius and height of the cylinder, and ν is the viscosity of the fluid.

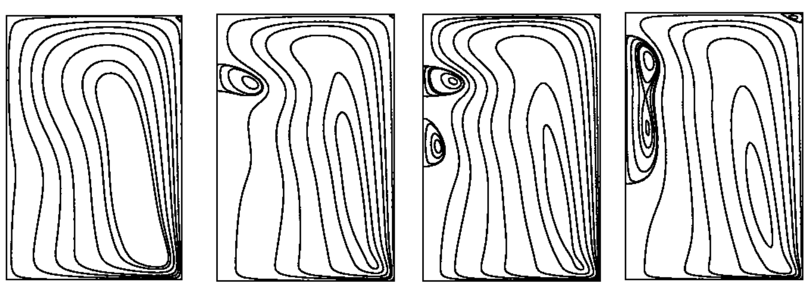

Figure 1: Typical topologies of iso-curves of ψ in a meridional plane for different combinations of the parameters (1). The line to the left is the cylinder axis.

To analyze the creation and interaction of the breakdown bubbles as the system parameters are varied, a topological approach has turned out to be very useful. From axisymmetry

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and incompressibility it follows that a streamfunction $\psi(r,z)$ (where $r \geq 0$ is the radial variable and z is the axial variable) exists such that the intersection between the streamsurfaces winding around the axis is given by iso-curves of ψ. Using bifurcation theory, it is possible to classify possible changes in the flow patterns. For previous applications to vortex breakdown, see e.g. [2].

The purpose of the present paper is to perform a bifurcation analysis in dependence of the three system parameters in the region where the flow is steady, and for small values of the rotation ratio γ. We develop a systematic way of analyzing numerical simulations in terms of bifurcation theory.

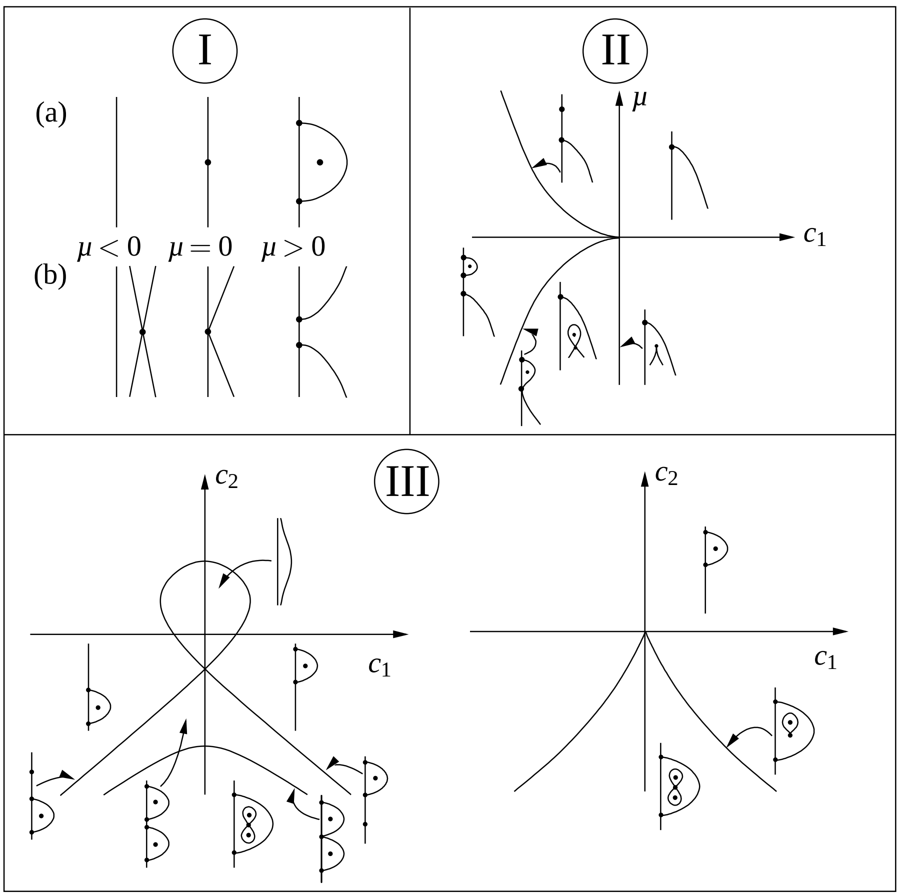

Figure 2: Abstract bifurcation diagrams. As in fig. 1, the vertical line on the left is the axis. Panel I: Codimension one bifurcations. Case (a) is bubble creation, case (b) is bubble merging. Panel II: Codimension two bifurcation. Panel III: Typical slices in a codimension three bifurcation diagram.

Mathematical preliminaries

The creation of new flow patterns is governed by local properties of the streamfunction. When certain combinations of derivatives of ψ are zero at a point, the flow pattern may be *degenerate* or *structurally unstable*. Here, arbitrarily small changes of ψ may give rise to changes in the patterns. A degeneracy is associated with an integer, the *codimension*, which essentially measures the number of degeneracy conditions which are fulfilled. A degeneracy of codimension n will generically occur only in systems with n free parameters, as the determination of such a point may be found by considering the degeneracy conditions as equations in the n free parameters.

From the theory of normal forms, diagrams for bifurcations close to the rotation axis can be obtained [1], the result is shown in fig. 2. The parameters μ, c_i shown in fig. 2 are formal mathematical quantities which can be expressed as complicated functions of the derivatives of ψ. Here we consider a regime where there is a unique steady state. In such a case, the mathematical parameters are functions of the physical parameters. The theory also gives the degeneracy conditions: With $w = -\partial\psi/\partial r$ denoting the axial velocity, a degeneracy of codimension n occurs at a point on the axis, say $(r,z) = (0,0)$, if

$$w(0,0) = 0, \quad \frac{\partial^k w}{\partial z^k}(0,0) = 0, \; k = 1 \ldots n-1, \quad \frac{\partial^n w}{\partial z^n}(0,0) \neq 0. \tag{2}$$

Numerical method and results

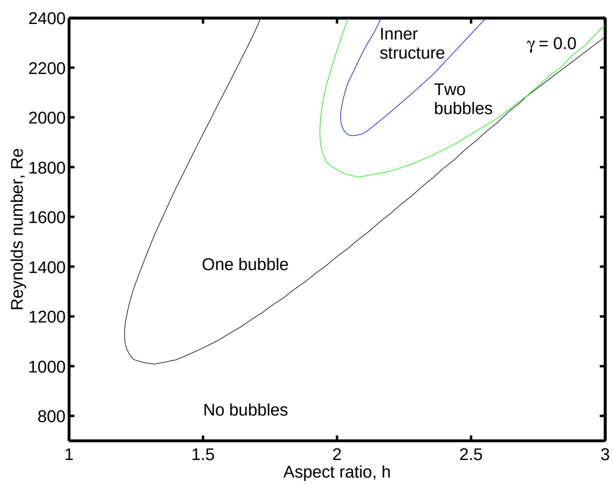

Figure 3: Bifurcation diagram for $\gamma = 0$.

Using the finite-difference method developed in [6] numerical simulations are performed on a (h, Re) grid for fixed values of γ. The grid spacing is typically $\Delta h = 0.05$ and $\Delta Re = 15$. Simulations are performed until a steady state has been reached. For this state, the axial velocity at the axis $w_a(z) = w(0,z)$ is extracted. Let $z_f(h, Re)$ denote a zero of the derivative, $w'_a(z_f(h,Re)) = 0$ and consider the function $w_a(z_f(h,Re))$ on the parameter plane. Letting Matlab draw the solution curve in the parameter plane to $w_a(z_f(h,Re)) = 0$, a curve of codimension one points is determined, according to eq. (2). Typically there are several zeroes of $w_a(z)$, and each must be treated separately, as they correspond to bifurcations different places on the cylinder axis. The two kinds of bifurcations, bubble creation (a) and bubble merging (b) can in principle be distinguished by computing the sign of w''_a at the degeneracy. For practical purposes, however, a visual inspection of sample simulations has been found to be sufficient.

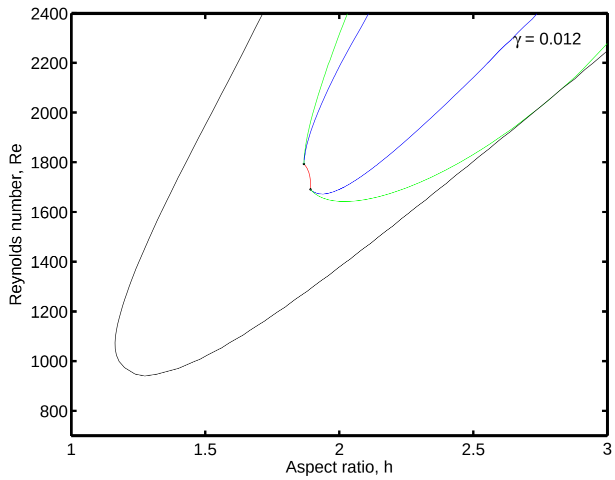

Figure 4: Bifurcation diagram for $\gamma = 0.012$.

Proceeding in this way, the bifurcation diagram for $\gamma = 0$ shown in fig. 3 is constructed. The lower curve marks the creation of a breakdown bubble, and the middle curve is the creation of a second bubble at a lower point on the axis. The two bubbles may merge in a type (b) bifurcation, creating a single large bubble with an inner figure-eight structure. The latter bifurcation takes place at the highest curve.

Increasing γ, the two bubble creation curves approach each other, and at $\gamma_1 = 0.0110$ they meet at a single point (h_1, Re_1). Increasing γ slightly beyond γ_1, a bifurcation diagram like fig. 4 occurs. In this diagram, two points of codimension two appear, where three bifurcation curves meet. The local structure of the bifurcations close to these points are as indicated in fig. 2 II. The bifurcation diagram includes a bifurcation curve of codimension one bifurcations off the cylinder axis, namely the merging of a saddle and a center in a saddle-node bifurcation. This bifurcation curve join the two codimension points. The numerical procedure for locating this curve differs from the procedure for locating bifurcations at the cylinder axis and has not been automated to the same extent. The details will appear elsewhere [3].

The change in the bifurcation diagram at γ_1 can be understood as a codimension three phenomenon. For a fixed value of γ, the physical (h, Re) plane is mapped to the mathemat-

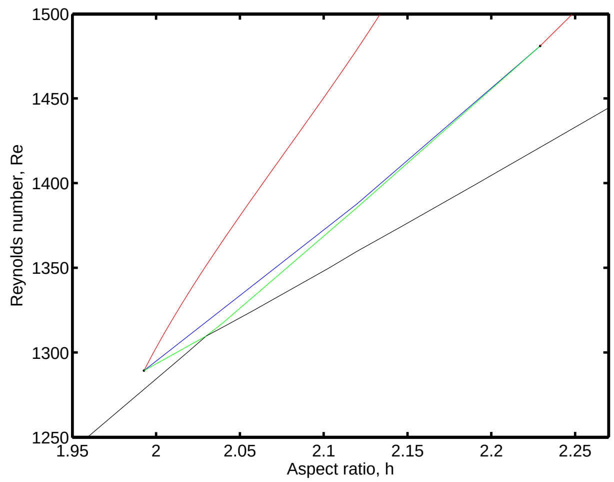

Figure 5: Detail of the bifurcation diagram for $\gamma = 0.0675$. A new codimension two point appears in the right top corner where two bifurcation curves merge.

ical (c_1, μ) plane of fig. 2 II. If this mapping folds the physical plane into the mathematical plane, it is not difficult to show that a generic possibility for a sequence of bifurcation diagrams are indeed qualitatively like figs. 3 – 4. The extra degeneracy condition is that a derivative of a mathematical parameter with respect to a physical parameter is zero.

Increasing γ to 0.0675, a bifurcation diagram for which a detail is shown in fig. 5 occurs. The leftmost codimension two point from fig. 4 has moved upwards and out of the diagram, and two new codimension two points are created in a bifurcation we do not touch upon here, one of which appear in the top right corner of fig. 5. As γ is increased further, the codimension two points move together and touch the primary bubble creation curve at $\gamma_2 = 0.0750$. For higher values of γ, the bifurcation diagram typically looks like fig. 6. We have not performed simulations beyond this value of γ. The change happening at γ_2 is also a codimension three phenomenon, this time associated with the bifurcation diagram in fig. 2 III. It is possible to find a series of planes moving through the three-dimensional mathematical parameter space where the intersection with the bifurcation surfaces qualitatively match figs. 5 – 6, but this time with no folding.

Conclusions

The topological approach to patterns in fluids provides a theoretical framework to generate catalogs of possible streamline patterns and how they change. The concept of degeneracy and codimension give useful terms to understand the organization of bifurcation diagrams, and efficient tools for analysis of numerical data can be developed on the basis of the theory. In the present problem there are three free parameters, and hence bifurcations of codimension up to three are of interest. Indeed, we find at least two codimension three

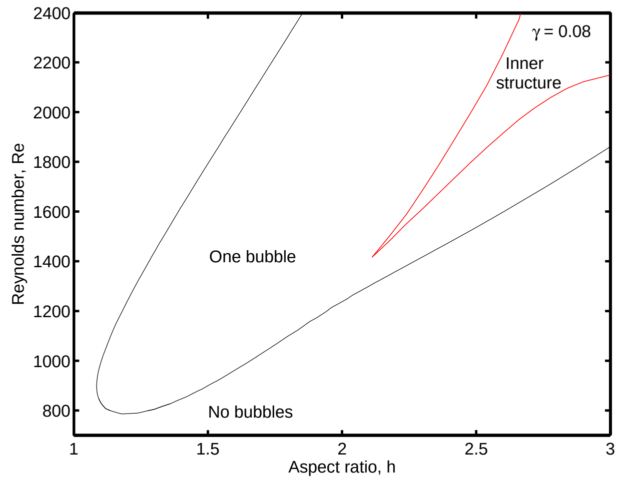

Figure 6: Bifurcation diagram for $\gamma = 0.08$.

points which organize the bifurcation diagrams. From a physical point of view, the sensitivity of the vortex breakdown to changes in the rotation rate is remarkable. A mathematical indicator of this is the presence of codimension three points very close to one another. A small change in the settings may move them together, creating a point of codimension four or higher. Thus, further additions to parameters governing changes in the set-up may give rise to new bifurcation phenomena.

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