A Model for Predicting the Evolution of Multiple Cracks on Multiple Length Scales in Heterogeneous Viscoelastic Solids

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Summary

A multiscale model is proposed herein for predicting the response of viscoelastic solids that undergo evolutionary cracking on multiple length scales. The model is formulated in such a way that analytic solutions may be utilized on some scales, and computational solutions may be utilized on others. The governing field equations may be rigorously derived on each length scale, and homogenization principles may be used to link the results obtained on each length scale. The implementation of the model to a hybrid analytical/computational algorithm is briefly described, with the finite element method employed on the largest two length scales. Finally, an example problem is solved in order to demonstrate both the accuracy and applicability of the model to a variety of current problems of engineering interest.

Introduction

A variety of viscoelastic solids undergo evolutionary cracking on multiple length scales. These cracks can interact and coalesce, sometimes leading to component failure. Applications where this can occur include geologic salt, nuclear weapons detonator materials, solid rocket propellant, tank armor, and asphaltic pavement. Currently utilized models for predicting the response of such media typically utilize a continuum damage mechanics model that accounts for the small scale damage in a phenomenological way at the macroscale. This approach, while it is indeed sometimes accurate, has some shortcomings. Among these are the necessity to perform cumbersome and costly experiments in order to characterize the macroscale constitutive behavior of the material, as well as the loss of the ability to account for the effects of the microscale variables, so that design considerations cannot account for such important effects as volume fractions of microscale additives. Multiscale models that are capable of predicting the response of viscoelastic media to both long term and impact loadings can potentially resolve both of these problems, thereby improving design procedures and decreasing costs simultaneously, while at the same time enhancing reliability.

In this paper a multiscale model will be outlined for predicting the response of such media to a variety of loading conditions. The model is formulated in such a way that analytic solutions may be utilized on some scales, and computational procedures may be utilized on others. The evolution of cracks on multiple scales is accounted for via the incorporation of a new nonlinear viscoelastic micromechanically based cohesive zone model. The implementation of the model to a hybrid analytical/computational algorithm is briefly described, with the finite element method employed on the largest two length scales. The following section outlines the model development.

Multiscale Model Development

Consider a solid region that may be heterogeneous, and has microstructure due to separate phases, grain boundaries, internal flaws, and/or damage. It is assumed that the length

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scale associated with the microstructure is significantly smaller than the length scale of the macroscale; that is,

$$l^{\mu} << l^{\mu+1} \qquad \mu = 1,\ldots,n \tag{1}$$

where l is the characteristic length for the particular scale of interest, the superscript refers to the scale number, and n represents the number of length scales observed in the problem of interest. While we will concentrate on only two scales in the present scenario, it is not uncommon to observe several length scales in practical problems. Indeed, this approach will be applicable to any number n so long as the object of interest may be accurately modeled as a continuum on each length scale.

Consider now a two scale example, as shown in Fig. 1, where the smaller scale is defined by a representative volume element (RVE) whose dimensions must be small compared to the global scale, but large enough to ensure statistical homogeneity of the state variables within the RVE. Mathematically, this implies that within the RVE

$$\overline{\theta}^{\mu} << SD(\theta^{\mu}) \tag{2}$$

where θ^{μ} is a generic state variable (such as stress, strain, and displacement) at the smaller length scale,

$$\overline{\theta}^{\mu} \equiv \frac{1}{V^{\mu}} \int_{V^{\mu}} \theta dV \tag{3}$$

is the volume average of the generic state variable in the RVE, and SD{.} is the standard deviation of the generic state variable, given by

$$SD\left\{\theta^{\mu}\right\} \equiv \left[\frac{1}{V^{\mu}} \int_{V^{\mu}} \left(\overline{\theta}^{\mu} - \theta^{\mu}\right)^2 dV \right]^{1/2} \tag{4}$$

The implication of inequality (2) above is that for purposes of modeling the response of the medium at the macroscale, the volume average of the generic state variable carries sufficient information about the state within the RVE, so that the full field of this variable is not required at the next larger scale. Thus, localization for example cannot be accurately captured by the use of the mean values indicated above.

Our intention is to solve two separate initial boundary value problems for these two scales, as opposed to a single one for both simultaneously. These two separate solutions are to be linked via a homogenization principle, and the intended objective is to gain computational and/or analytic simplicity without sacrificing significant accuracy. The reasons that this approach has potential are as follows [Allen, 2002]:

1) solving the complete problem will normally be untenable when numerous cracks of widely varying dimensions are present since large numbers of finite elements are necessary for each individual crack;

2) the mean values of the state variables at the smaller scale may have a profound impact on the accuracy of the model predictions at the next larger scale; and
3) the construction of a linked multiscale model produces a superior model to a single global model in that the details that are accounted for at the smaller length scales can be incorporated as design parameters affecting the performance at the global scale.

The two stage solution approach is initiated by first posing the initial boundary value problem for each representative volume element on the local scale. This local problem is then solved by any means at hand, i.e., analytically if such a solution exists, or numerically (such as the finite element method) if an analytic solution does not exist.

Assuming that the solution to the local problem has been obtained for each representative volume element in the global scale problem, the next step is to link the local scale to the global scale. This is accomplished by defining boundary averages of the state variables at the local scale. Thus, the boundary averaged displacements are defined by

$$\hat{u}_i^\mu \equiv \frac{1}{V^\mu} \int_{\partial V^\mu} u_i^\mu dS \tag{5}$$

where u_i^μ is the displacement vector on the smaller scale, and V^μ is the volume of a representative volume element (RVE), defined by inequality (2). Furthermore, the boundary averaged stresses are defined as

$$\hat{\sigma}_{ij}^\mu \equiv \frac{1}{V^\mu} \int_{\partial V_E^\mu} \sigma_{ik}^\mu n_k^\mu x_j^\mu dS \tag{6}$$

where $\vec{n}^\mu$ is the unit outer normal vector to the external undamaged boundary ∂V_E^μ of the RVE, and σ_{ik}^μ is the stress tensor on the smaller scale. Finally, the boundary averaged strain is defined by

$$\hat{\varepsilon}_{ij}^\mu \equiv \frac{1}{V^\mu} \int_{\partial V_E^\mu} \frac{1}{2}(u_i^\mu n_j^\mu + u_j^\mu n_i^\mu) dS \tag{7}$$

By adding the internal boundary term to equation (6) and employing the divergence theorem and Cauchy's formula, it can be shown that in the absence of inertial effects and body forces (by deploying conservation of linear momentum):

$$\bar{\sigma}_{ij}^\mu = \hat{\sigma}_{ij}^\mu + \hat{\varsigma}_{ij}^\mu \tag{8}$$

where

$$\hat{\varsigma}_{ij}^\mu \equiv \frac{1}{V^\mu} \int_{\partial V_I^\mu} t_i^\mu dS \tag{9}$$

is the boundary average of the tractions, t_i^μ, on the crack faces and/or internal boundaries ∂V_I^μ. In the case wherein the internal boundaries are all produced by crack extension, it can reasonably be assumed that the crack face tractions are self equilibrating, so that $\hat{\varsigma}_{ij}^{\mu} = 0$, with the necessary conclusion that

$$\hat{\sigma}_{ij}^{\mu} = \bar{\sigma}_{ij}^{\mu} \tag{10}$$

In other words, when there are only internal cracks, the external boundary averaged stress is equivalent to the volume averaged stress in the RVE.

We assume herein that the difference in length scales is great enough that inertial effects and body forces can always be neglected at all scales except the global scale, so that equation (10) is true for all values of $\mu < n_l$, where n_l is the number of scales considered in the solution scheme. A similar procedure will result in the following kinematic equations:

$$\bar{\varepsilon}_{ij}^{\mu} = \hat{\varepsilon}_{ij}^{\mu} + \hat{\alpha}_{ij}^{\mu} \tag{11}$$

where [Eshelby, 1957]

$$\hat{\alpha}_{ij}^{\mu} \equiv \frac{1}{V^{\mu}} \int_{\partial V_I^{\mu}} \frac{1}{2}(u_i^{\mu} n_j^{\mu} + u_j^{\mu} n_i^{\mu}) dS \tag{12}$$

as we noted earlier, it is in general too computationally cumbersome to utilize the full- field stresses on the larger scale from the next smaller length scale, σ_{ij}^{μ}, in the above. Our preference would be to obtain an approximate yet reasonably accurate simplification of the global problem that does not require full field values at the smaller scale, but rather employs a simpler measure such as the mean value of the stress from the smaller scale. Such a statement may be derived from the exact equations by integrating them over the volume of the RVE, and then employing the divergence theorem, together with the above boundary averaged variables, to produce a global initial value problem that is identical to the local boundary value problem solved for each local RVE, with one exception. That exception is that the global constitutive equations will be complicated by the fact that an additional damage dependent term appears. Thus, for example, for the case of linear viscoelastic material behavior at the microscale, the resulting constitutive equations at the macroscale will take on the following form [Searcy, 2004]:

$$\sigma_{ij}^{\mu+1}(t) = \int_0^t C_{ijkl}^{\mu}(t-\tau) \frac{\partial \varepsilon_{kl}^{\mu}}{\partial \tau} d\tau + (\sigma_{ij}^{D})^{\mu+1} \tag{13}$$

where $C_{ijkl}^{\mu}(t)$ is the linear viscoelastic relaxation modulus tensor on the smaller scale, and ε_{kl}^{μ} is the strain tensor on the smaller scale. Also,

$$\left(\sigma_{ij}^{D}\right)^{\mu+1} \equiv \frac{1}{V^{\mu}} \int_{\partial V_I^{\mu}} \left\{ \int_0^t C_{ijkl}^{\mu}(t-\tau) \frac{\partial u_k^{\mu}}{\partial \tau} n_l^{\mu} d\tau \right\} dS \tag{14}$$

Thus, with the exception of the last term in equation (13), the form of the global initial boundary value problem is identical to the local one. Most importantly, this last term can be calculated from the local solution for each RVE at each increment in time, so that equation (14) is known at each point in the global problem, and the global problem can then be solved with essentially the same algorithm that is utilized to solve the local one. Note that all damage in the global solution thus enters through the boundary integral reflected in equation (14). Thus, the technique for constructing a multiscale algorithm is well defined and the conditions under which it will be produce accurate predictions are established by the condition (2) above for all length scales of interest.

Development of a Multiscale Algorithm

The methodology described in the previous section has been successfully implemented to the finite element program SADISTIC [Allen et al., 1994, Searcy, 2004]. This algorithm can be used to model the evolution of cracks in viscoelastic media by utilizing a micromechanically based cohesive zone model previously developed by the authors [Allen and Searcy, 2001a, b]. This cohesive zone model can be deployed on any and all length scales for which crack evolution is expected to occur. The evolution of such cracking on any scale will naturally lead to the development of non-null values of $\sigma_{ij}^{D\mu+1}$ wherever cracking occurs within an RVE, and this term will then modify to the macroscale response of the medium. This procedure has been utilized to develop the multiscale version of SADISTIC deployed herein.

Sample Problem

The multiscale method briefly described in the previous two sections can be employed to predict the response of viscoelastic media that undergo multiscale cracking. As an example, we consider herein the case of a tapered uniaxial bar composed of a composite material with microstructure, as shown in Fig. 2. The bar is subject to uniaxial monotonically increasing loading, thus producing crack growth near the narrow end of the bar. The type of response shown in Fig. 3 is at least in qualitative agreement with experimental evidence.

Conclusion

A model has been briefly reviewed herein for predicting the evolution of cracks on multiple scales in heterogeneous viscoelastic media. An example problem has demonstrated the potential viability of this approach. However, more study is warranted before this model can be verified.

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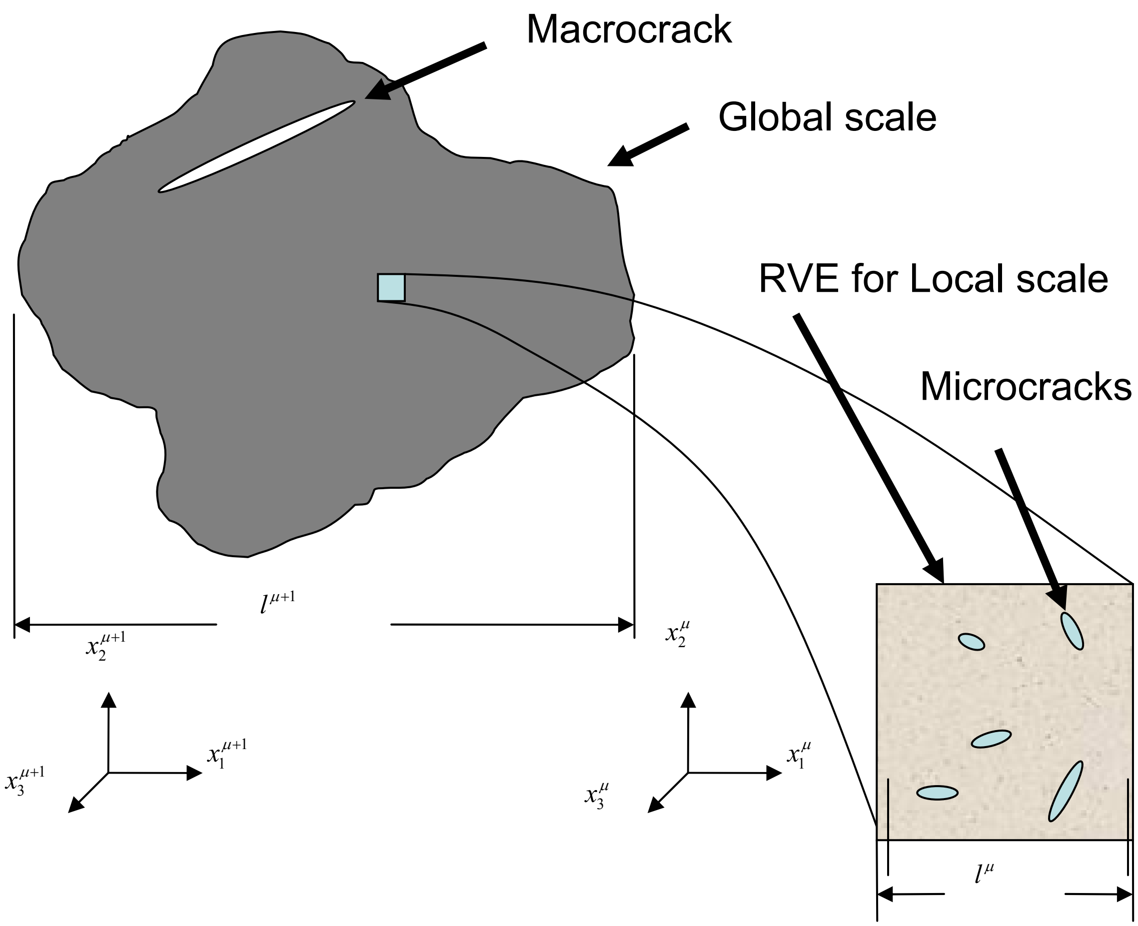

Fig. 1. Representation of a Continuum with Two Length Scales Due to Damage Accumulation

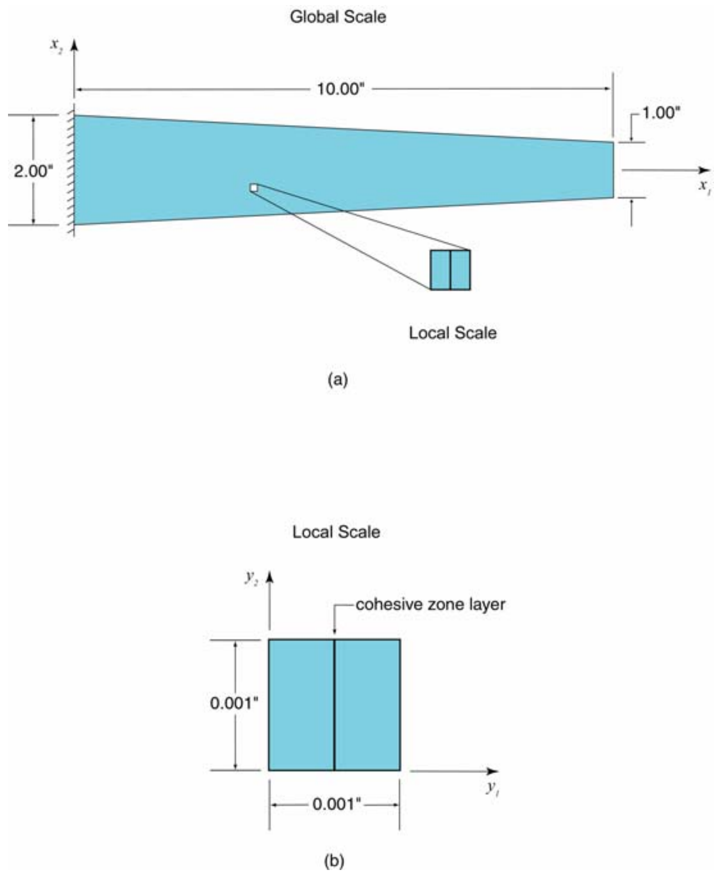

Fig. 2. Tapered Bar Geometries. (a) Global Geometry (b) Unit Cell.

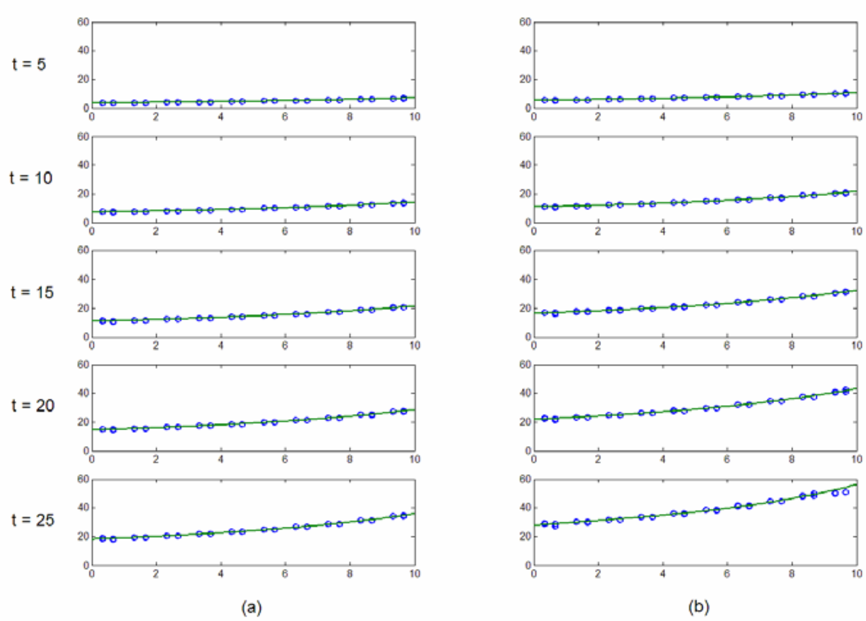

Fig.3. Axial Stress Distribution for Tapered Bar Problem (a) Without Multiscaling and (b) With Multiscaling.