# THE INFLUENCE OF IMPERFECTIONS ON THE NONLINEAR VIBRATIONS OF CYLINDRICAL SHELLS SUBJECTED TO INTERNAL FLOWING FLUID AND COMBINED LOADS

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## ABSTRACT

*The extensive use of thin cylindrical shells in different areas of engineering makes their proper design a very important subject. One of the most relevant topics is the study of their nonlinear behaviour when subjected to static and dynamic loads. Most of the studies on nonlinear vibrations are focused on the analysis of perfect shells, and only a few are directed to the analysis of the influence of geometric imperfections. In this paper, the non-linear vibrations of simply supported perfect and imperfect circular cylindrical shells with internal flowing fluid and subjected to external harmonic pressure are discussed. Attention is given to frequency-amplitude relation and to resonance curves.*

## 1. INTRODUCTION

The buckling behaviour of cylindrical shells is a central topic in the theory of elastic stability. Its simple geometry and its efficiency as a load carrying member, particularly for axial loads and lateral pressure, makes cylindrical shells one of the most common shell geometries in industrial applications and in nature. Applications can be found in different engineering branches such as civil, mechanical, nuclear and off-shore engineering. In most of these applications cylindrical shells are used to hold or transport fluids. Thus, the study of shell-fluid interaction considering quiescent or flowing fluid has been an important research area in applied mechanics (Amabili and Païdoussis, 2003).

It is a well established fact, backed by a large collection of theoretical and experimental results, that cylindrical shells under several types of static load, such as axial compression, external pressure, torsion and bending, are liable to buckling and may display a load capacity much lower than the theoretical critical load, due mainly to the effect of imperfections (Yamaki, 1984).

Cylindrical shells display unstable post-critical behaviour and are sensitive to the presence of small initial imperfections. Geometric imperfections caused by manufacturing are considered to be the main cause of the significant differences between critical buckling loads calculated using classical methods and experimental buckling loads (Batista and Gonçalves, 1994). This is mainly due to the highly nonlinear behaviour of cylindrical shells and the reduction of membrane stiffness, modal coupling and interactions generated by the imperfections. Experimental results on thin cylindrical shells under static axial compression show that they may lose stability at load levels as low as a fraction of the material's ultimate strength (Yamaki, 1984). However, the influence of these geometric imperfections on the vibration characteristics and nonlinear oscillations of thin shells is not well understood. In a dynamic environment, several parameters may influence the imperfection sensitivity, such as initial imperfections and load characteristics. The addition of fluid forces makes this problem even more complicated. Also, changes in the effective stiffness due to the static pre-loading may affect drastically the dynamic response and stability characteristics of these systems (Gonçalves and Del Prado, 2002; Gonçalves et al., 2007a,b).

The study of the dynamics of cylindrical shells is one of the most challenging problems in nonlinear dynamics. Although geometrical imperfections play an important role in the shell response (Amabili, 2003; Pellicano and Amabili, 2006; Catellani et al. 2004), the great majority of the investigations are related with the study of perfect structures. Fewer studies are focused on the analysis of the nonlinear vibrations of cylindrical shells in contact with a dense fluid.

An extensive literature review related to the non-linear dynamics of shells can be found in Amabili and Païdoussis (2003) and Karagiozis (2006).

Watawala and Nash (1983) studied the influence of initial geometric imperfections on free vibrations of fluid filled cylindrical shells subject to seismic motions of the base. Amabili and Pellicano (2002) studied the nonlinear stability of simply supported, circular cylindrical shells in supersonic axial flow using Donnell's nonlinear shallow shell theory, considering asymmetric and axisymmetric geometric imperfections. Amabili (2003), using an accurate mode expansion, studied the large-amplitude response of perfect and imperfect, simply supported circular cylindrical shells subjected to harmonic excitation in the neighbourhood of some of the lowest natural frequencies. Catellani et al. (2004) analyzed the static and dynamic behaviour of a compressed circular cylindrical shell with geometric imperfections using Donnell's nonlinear shallow-shell theory. Pellicano and Amabili (2006), using both Donnell's nonlinear shallow shell and the Sanders–Koiter shell theories, analyzed the dynamic stability of cylindrical shells in the presence of static and dynamic axial loads, geometric imperfections and fluid–structure interaction.

In this work, a five degree-of-freedom model is used to study the influence of initial imperfections on the nonlinear oscillations and instabilities of cylindrical shells with internal flowing fluid. The fluid is assumed to be incompressible and inviscid and the flow to be isentropic and irrotational. The shell is subjected to a harmonic radial pressure load. The Donnell shallow shell equations are used to describe the shell and used together with the Galerkin method to derive a set of coupled ordinary differential equations of motion. In order to study the nonlinear behaviour of the shell, several numerical strategies were used to obtain Poincaré maps, stability boundaries and bifurcation diagrams.

## 2. PROBLEM FORMULATION

### 2.1 Governing equations

Consider a simply supported thin-walled cylindrical shell of radius $R$, length $L$ and thickness $h$. The shell is assumed to be made of an elastic, homogeneous and isotropic material with Young's modulus $E$, Poisson ratio $\nu$ and density $\rho_s$.

The radial, circumferential and axial co-ordinates are denoted by $z$, $y \equiv R\theta$ and $x$, respectively, and the corresponding displacements of the shell middle surface are in turn denoted by $w$, $v$ and $u$. In this work the mathematical formulation will follow that previously presented in references (Amabili et al., 1999; Pellicano and Amabili, 2006; Pellicano et al., 2002; Catellani et al., 2004). The shell is subjected to an internal flowing fluid, a static axial load ($\tilde{N}_x$) and a harmonic external pressure given by $f = f_o \cos(n\theta)\sin(\pi x/L)\cos(\omega_L t)$.

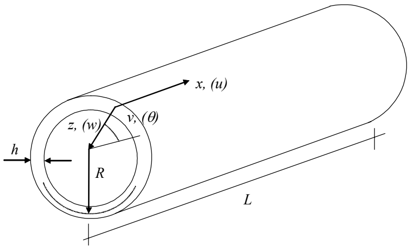



*Figure 1: Shell geometry*

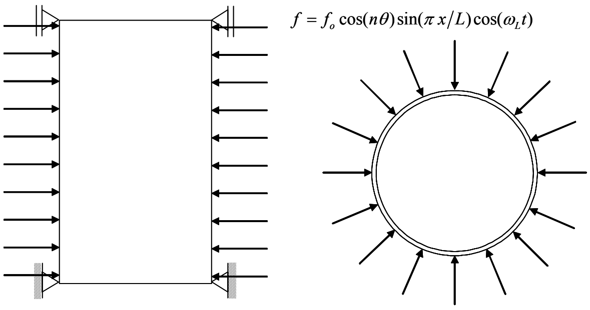



*Figure 2: Harmonic external pressure*

The nonlinear equation of motion, based on Donnell's shallow shell theory, in terms of a stress function $F$, the transverse displacements $w$ and the initial geometric imperfection field $w_o$ is given by

$$D\nabla^4 w + c\,h\,\dot{w} + \rho_s\,h\,\ddot{w} = f - P_h + \frac{1}{R}\frac{\partial^2 F}{\partial x^2} + \frac{1}{R}\left[\frac{\partial^2 F}{\partial \theta^2}\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w_o}{\partial x^2}\right) - 2\frac{\partial^2 F}{\partial x\,\partial \theta}\left(\frac{\partial^2 w}{\partial x\,\partial \theta} + \frac{\partial^2 w_o}{\partial x\,\partial \theta}\right) + \frac{\partial^2 F}{\partial x^2}\left(\frac{\partial^2 w}{\partial \theta^2} + \frac{\partial^2 w_o}{\partial \theta^2}\right)\right], \quad (1)$$

where $P_h$ is the fluid flow perturbation pressure, $c$ (kg/m$^3$ s) is the damping coefficient and $D$ the flexural rigidity defined as $D = Eh^3/[12(1-\nu^2)]$.

The compatibility equation is given by

$$\frac{1}{Eh}\nabla^4 F = -\frac{1}{R}\frac{\partial^2 w}{\partial x^2} - \frac{1}{R^2}\left[ -\left(\frac{\partial w}{\partial x \partial \theta}\right)^2 \right.$$

$$-2\frac{\partial^2 w}{\partial x\,\partial \theta}\frac{\partial^2 w_o}{\partial x\,\partial \theta} + \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w_o}{\partial x^2}\right)\frac{\partial^2 w}{\partial \theta^2}$$

$$\left. + \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w_o}{\partial \theta^2}\right]. \qquad (2)$$

In equations (1) and (2) the biharmonic operator is defined as $\nabla^4 = [\partial^2/\partial x^2 + \partial^2/(R^2 \partial \theta^2)]^2$.

## 2.2 Fluid loading

The shell is assumed to be filled with a fluid flowing with velocity $U$. To determine the perturbation pressure on the shell wall, the Païdoussis and Denise (2004) model will be adopted. In this model, linear potential theory is used to describe the effect of the internal axially flowing fluid. The fluid is assumed to be incompressible and inviscid and the flow to be isentropic and irrotational. The irrotationality property is the condition for the existence of a scalar potential function $\Psi$, from which the velocity may be written as

$$V = -\nabla \Psi . \qquad (3)$$

This potential function is equal to $\Psi = -U\,x + \Phi$, where the first term is associated with the undisturbed mean flow velocity $U$, and the second, the unsteady component $\Phi$ is associated with shell motion. The function $\Phi$ must satisfy the Laplace equation and the impenetrability condition at the shell-fluid interface.

The potential function satisfies the continuity equation. Following the procedure presented in previous studies (Amabili et al., 1999; Païdoussis, 2004) the perturbation pressure on the shell wall is found to be

$$P_h = \rho_F \frac{L}{m\pi}\frac{\mathrm{I}_n}{\mathrm{I}_n^{'}}\left(\frac{\partial^2 w}{\partial t^2} + 2U\frac{\partial^2 w}{\partial t\,\partial x} + U^2\frac{\partial^2 w}{\partial x^2}\right), \qquad (4)$$

where $\mathrm{I}_n$ is a modified Bessel function, $\mathrm{I}_n^{'}$ its derivative with respect to the argument and $\rho_F$ is the fluid density.

## 2.3 Modal expansion

The numerical model is developed by expanding the transverse displacement component $w$ in a series in the circumferential and axial variables. The following mode expansion with 5 degrees-of-freedom, which satisfies the out-of-plane simply-supported boundary conditions, is used:

$$w = \xi_{1,1} h \sin\left(\frac{\pi x}{L}\right)\cos(n\theta) + \xi_{0,1} h \sin\left(\frac{\pi x}{L}\right) +$$

$$\xi_{0,3} h \sin\left(\frac{3\pi x}{L}\right) + \xi_{0,5} h \sin\left(\frac{5\pi x}{L}\right) +$$

$$\xi_{0,7} h \sin\left(\frac{7\pi x}{L}\right), \qquad (4)$$

where $\xi_i$ are the modal amplitudes and $n$ is the circumferential wave number.

The geometric initial imperfection of the shell is modelled using the same expansion as for the lateral displacement, that is

$$w_o = \Xi_{1,1} h \sin\left(\frac{\pi x}{L}\right)\cos(n\theta) + \Xi_{0,1} h \sin\left(\frac{\pi x}{L}\right)$$

$$+ \Xi_{0,3} h \sin\left(\frac{3\pi x}{L}\right) + \Xi_{0,5} h \sin\left(\frac{5\pi x}{L}\right)$$

$$+ \Xi_{0,7} h \sin\left(\frac{7\pi x}{L}\right), \qquad (5)$$

where $\Xi_i$ are the imperfection modal amplitudes.

The solution for the stress function may be written as $F = F_h + F_p$, where $F_h$ is the homogeneous solution and $F_p$ the particular solution. The particular solution $F_p$ is obtained analytically by substituting the assumed form of the lateral deflection, $w$, Eq. (4), and the geometric imperfection, $w_o$, Eq (5), on the right-hand side of the compatibility equation, Eq. (2), and by solving the resulting linear partial differential equation together with the relevant boundary and continuity conditions.

The homogeneous part of the stress function can be written as

$$F_h = \frac{1}{2}\overline{N}_x R^2 \theta^2 + \frac{1}{2}x^2 \left\{ \overline{N}_\theta \right.$$

$$\left. - \frac{1}{2\pi R L}\int_0^{2\pi}\int_0^{L}\frac{\partial^2 F_p}{\partial x^2} R\, d\theta\, dx \right\} - \overline{N}_{x\theta} R\theta , \qquad (6)$$

where $\overline{N}_x$, $\overline{N}_\theta$ and $\overline{N}_{x\theta}$ are the average in-plane restraint stresses generated at the ends of the shell. This solution enables one to satisfy the in-plane boundary conditions on the average. Boundary conditions allow us to express the in-plane restraint stresses $\overline{N}_x$, $\overline{N}_\theta$ and $\overline{N}_{x\theta}$ in terms of $w$ and its derivatives (Amabili et al., 1999; Pellicano et al., 2002, Catellani et al., 2004); thus,

$$\overline{N}_x = \widetilde{N}_x ,$$

$$\overline{N}_\theta = \nu\,\widetilde{N}_x + \frac{E\,h}{2\pi L}\int_0^{2\pi}\int_0^{L}\left[-\frac{w}{R} + \frac{1}{2}\left(\frac{1}{R}\frac{\partial w}{\partial \theta}\right)^2\right] d\,x\, d\theta , \qquad (7)$$

$$\overline{N}_{x\theta} = 0 ,$$

where $\widetilde{N}_x$ is the axial load acting at the ends of the shell.

## 3. NUMERICAL RESULTS

Consider a thin-walled cylindrical shell with $h$=0.002 m, $R$=0.2 m, $L$=0.4 m, $E$=2.1x10$^8$ kN/m$^2$, ν=0.3, $\rho_S$=7850 kg/m$^3$ and $\rho_F$ =1000 kg/m$^3$ (Gonçalves and Del Prado, 2002). For this geometry the critical flow velocity is $U_{cr}$ = 257.53 m/s. The lowest natural frequency of the fluid filled shell is $\omega_o$ = 1704.33 rad/s, which occurs for $(n,m)$ = (5,1). The damping coefficient is defined as c = $2\zeta\rho_S\omega$. In the present analysis, the damping coefficient adopted is $\zeta$ = 0.089.

Figure 3 shows the post-critical paths of the shell with no flowing fluid and varying imperfections on first mode. The amplitude of lateral pressure is 20% of the lateral critical load. Comparing the post-critical paths with that of the perfect shell, it can be seen that lateral pressure, together with increasing imperfections, have a marked reduction on the value of the limit load of the shell.

Figure 4 displays the post-critical paths for increasing values of flowing fluid for a shell loaded with 20% of the lateral critical load and imperfection on first mode equal to $\Xi_{1,1}$=0.1. In both Figures 3 and 4, imperfections and fluid flow generate a reduction of the limit loads, also reducing the value of the hardening starts.

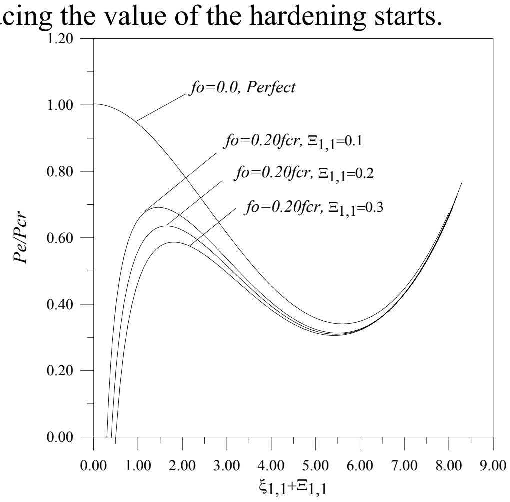



*Figure 3: Post-critical path for varying imperfection and U=0.0*

Figure 5 shows the variation of the nonlinear free vibration amplitude with the variation of the nonlinear natural frequency for a perfect and imperfect fluid filled shell with no axial load. In Figure 5(a), initial geometric imperfections are considered only on the first mode ($\Xi_{1,1}$=0.1 and $\Xi_{1,1}$=0.2). By comparing the perfect and imperfect shell responses, it is possible to verify that small imperfections on the first mode have a negligible influence on the nonlinear natural frequency. However, the initial branch of the curves displays a more pronounced softening behaviour as the level of imperfection is increased.

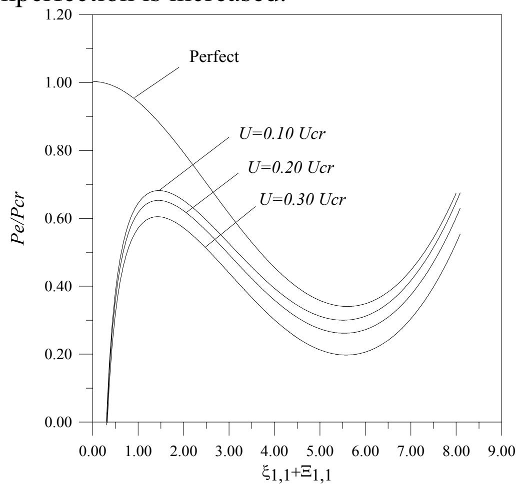
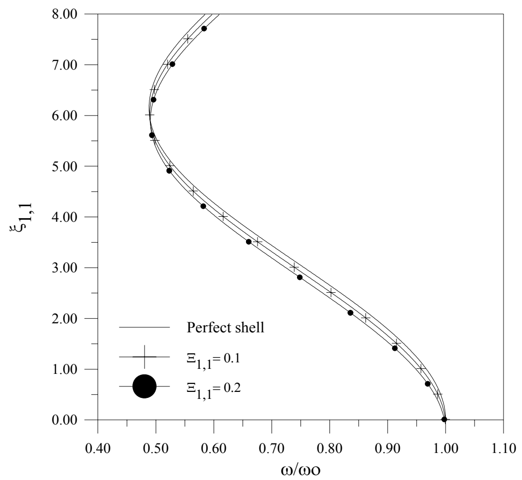



*Figure 4: Post-critical paths for varying fluid flow.*

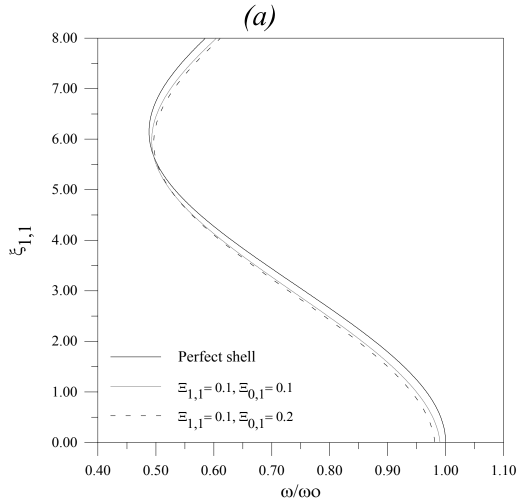
In Figure 5(b) imperfections are considered on both first and second modes ($\Xi_{1,1}$=0.1, $\Xi_{0,1}$=0.1 and $\Xi_{1,1}$=0.1, $\Xi_{0,1}$=0.2). In this case, the imperfections reduce the value of the nonlinear natural frequency. They also increase the degree of the softening behaviour. Figure 5(c) shows the effect of fluid flow for an imperfect shell ($\Xi_{1,1}$=0.1, $\Xi_{0,1}$=0.1). The fluid velocity has a marked influence both on the natural frequency and on the frequency-amplitude relation. For example, if $U$=0.20$U_{cr}$, the natural frequency of the shell is reduced by 3%, and the softening behaviour of the nonlinear response increases. For $U$=0.40$U_{cr}$, the shell displays a more strongly softening branch and the natural frequency is reduced in about 10%.

Figure 6 shows the resonance curve for a perfect fluid-filled shell. The amplitude of the lateral harmonic pressure $f_o$ is equal to 20% of the critical load of a shell subjected to a uniform external pressure. These figures are obtained through continuation techniques and the Newton-Raphson method. Here and ion the following, black lines represent stable branches and grey lines represent unstable branches. The shell shows a typical softening behaviour and results are in agreement with a previous work by Pellicano et al. (2002).

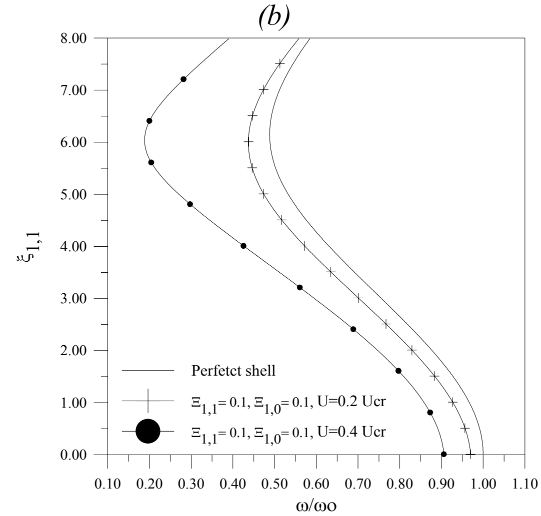
Figure 7 displays the resonance curve for a perfect shell. The amplitudes of the lateral harmonic pressure, static axial load and fluid flow are given as a fraction of the critical value given by $f_o = 0.20 f_{cr}$, $\tilde{N}_x = 0.40\tilde{N}_{x\,cr}$ and $U$=0.2$U_{cr}$. Here the shell shows a typical softening behavior with a shifting to the left of the curve if compared with Figure 6. In these curves, the shell shows an increase in the amplitude of vibrations and the possibility of a jump in the bifurcation points.




*(a)*




*(b)*




*(c)*

*Figure 5: Frequency-amplitude relation.*

Figure 8 depicts the resonance curve for increasing levels of imperfections. It shows the

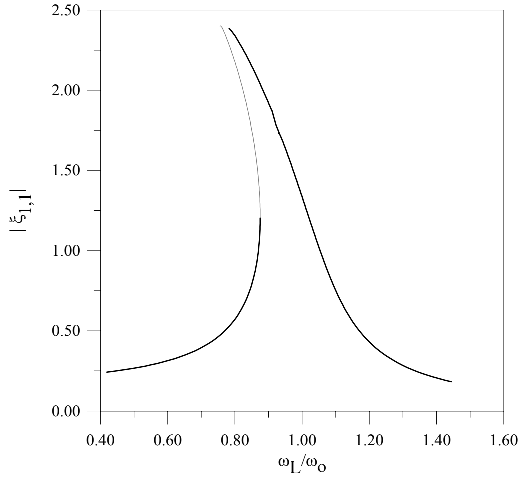



*Figure 6: Response-frequency curve for perfect shell with $f_o$=0.20$f_{cr}$, U=0.0.*

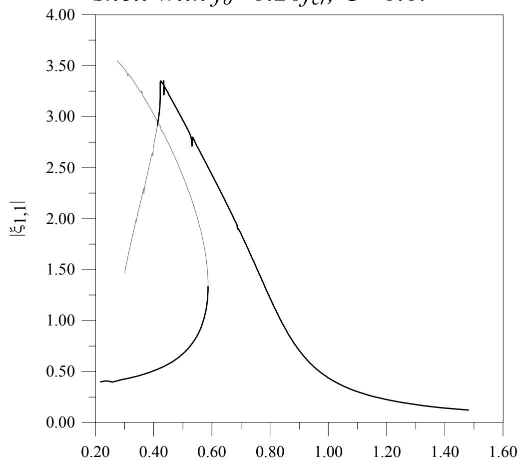



*Figure 7: Response-frequency curve for perfect shell with $f_o$=0.20$f_{cr}$, U=0.20$U_{cr}$,* $\tilde{N}_x = 0.40\tilde{N}_{x\,cr}$.

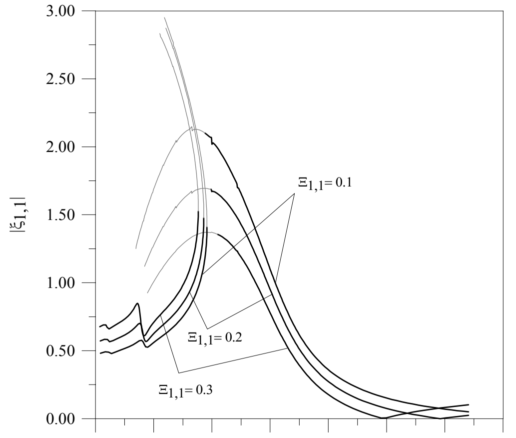



*Figure 8: Response-frequency curve for imperfect shell with $f_o$=0.20$f_{cr}$, U=0.20$U_{cr}$,* $\tilde{N}_x = 0.40\tilde{N}_{x\,cr}$.

strong influence of imperfections on the nonlinear vibrations of the shell. On the left branch of the curves, there is an increment of the amplitude of vibration and, on the right branch; there is a reduction of the amplitudes.

As the level of imperfection is increased, a window of unstable solutions appears. Since imperfections are not known *a priori* and may change during the service life of the structure, the outcome of the response at large amplitude vibration is uncertain.

## 4. CONCLUSION

Based on Donnell's shallow shell equations, a 5 dof model is applied to study the nonlinear vibrations of perfect and imperfect cylindrical shells under harmonic lateral pressure and fluid flow. Initial geometric imperfections are considered to have the same shape as the shell transverse displacement field.

The influence of geometric imperfections on the frequency-amplitude relation and on the resonance curve is studied. The results show that imperfections with components in the two first modes (nonlinear mode), produce a higher decrease of the nonlinear natural frequency than imperfections only on first mode. They also influence the stability of the large amplitude resonant response. The results also show the marked influence of the fluid flow velocity on the response of the shell, decreasing the natural frequency and increasing the shell softening behaviour.

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