SIMPLE EQUATIONS FOR ACOUSTIC STRUCTURAL COUPLING

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ABSTRACT

When an acoustic resonance excites a structural resonance the possibility exists for large and potentially damaging structural stresses. The prediction of stresses under these circumstances is complicated by the fact that the level of stress is sensitive to the precise tuning of the system. Consequently, a typical computer based simulation will probably miss the worst case while the built system may suffer failure because, by chance, it is tuned to the worst case.

The amplitudes of vibration for the worst possible case of an acoustic-structural system are investigated and formulated as simple equations. These equations use parameters that may be determined by considering the acoustic and structural characteristics of the system in separate simulations. This avoids the need to undertake detailed simulations of the fully coupled system.

1. INTRODUCTION

The combination of an acoustic and structural resonance can often be problematic. A particularly hazardous case is that of gas pipework where large noise levels can induce significant pipe vibration resulting in fatigue failures and an escape of flammable gas.

In general, it is necessary to use computer based simulation methods, for example the finite element method, when investigating an acoustic-structural system. The standard approach is to make separate models for the acoustic and structural systems. This is usually necessary because a fully coupled acoustic-structural system is too complex for current computer software. This difficulty will probably continue into the future even with improved computer power because the acoustic system is often physically large compared to the small components of the structural system which are vulnerable to vibration. For example, pipeline acoustics may depend on hundreds of meters of pipe while the component set into vibration may be a few metres of side branch pipework.

The output from the acoustic analysis of the systems is a set of acoustic natural frequencies and modes shapes that assumes the structural system is rigid. The output from the structural system is a set of structural natural frequencies and mode shapes which assumes that the structure is flexible but that no fluid is present. The problem following such analyses is to predict the response of the combined acoustic-structural system.

As will be shown a coupled acoustic-structural system can have a wide range of resonant vibration amplitudes depending on the precise nature of the tuning between the two systems. Consequently an assessment procedure must not only work out the consequences of coupling the acoustic and structural systems but must also consider the effect of various tuning options that cover all practical possibilities.

Computer simulations are always inaccurate because, (i) the extent of the system is not modelled due to difficulties in the representation of boundary conditions; (ii) the system may be constructed differentially to the model, (iii) the system may be modified during its life or the excitation may change and finally (iv) all the details of the system are not fully included in the model. An approach is therefore needed which enables the worst possible circumstances to be determined even when basic parameters such as mass, stiffness or boundary conditions are not modelled accurately.

The solution to this modelling problem is to assume from the outset that the most unfavourable conditions will prevail and that not only will the system be excited in resonance but that it will be tuned to the worst possible resonant condition.

The objective of this work is thus to find the conditions that lead to the largest possible resonance amplitudes and then to find an equation for this amplitude in terms of basic structural parameters that can be deduced from elementary modelling.

2. PROBLEM FORMULATION

It is assumed that acoustic and structural modelling has been undertaken and that natural frequencies and mode shapes have been calculated separately for each system. If the natural frequencies of both systems are examined there will be immediate cause for concern when one acoustic natural frequency is similar to a structural natural frequency. This is the starting point for further analysis which must determine if the acoustic and structural natural frequencies will couple to produce high stress.

The approach taken here is to extract one acoustic and one structural mode and to examine the combined response of the coupled system. As an example of this process the configuration in Figure 1 is analysed. This configuration consists of an acoustic cavity of length L and cross-sectional area S coupled to a piston-like mass-spring-damper system at the right-hand end and driven by an acoustic source, with a harmonic volume velocity amplitude of Q, at the left-hand end Although this appears as a very specific example it may be generalised to include any case of acoustic structure interaction. One key advantage of this problem is that a closed form solution may be established.

Starting with the acoustic equations it is straightforward to model the pressure within the cavity by assuming plane waves and modelling the piston as a second source at the right hand end. This leads to an expression for the pressure, P, at the piston of

$$P = \frac{-i\rho c}{S} \frac{Q + Q_p\left(\cos\left(\frac{\omega L}{c}\right) + i\sin\left(\frac{\omega L}{c}\right)\right)}{\sin\left(\frac{\omega L}{c}\right) - i\alpha\cos\left(\frac{\omega L}{c}\right)} \quad (1)$$

where ρ is the fluid density, c the speed of sound, S the cross-sectional area, Q the volume velocity of the source on the left, Q_p the volume velocity of the source on the right that models the piston motion, ω the frequency of oscillation in radians per second, L the length of the cavity and α the ratio of the cross-sectional area of the semi-infinite pipe on the left to the area of the cross-section of the main cavity. This is a closed form solution but may be turned into a modal solution by finding the roots of the denominator and the residue of the function at these roots. The roots are given by

$$\omega_n = \frac{nc\pi}{L} + i\frac{c}{2L}\log\left(\frac{1+\alpha}{1-\alpha}\right) \quad (2)$$

where ω_n is the n^{th} natural frequency. The small pipe, of infinite extent going to the left in Figure 1 acts as a damper since waves propagate away from the cavity and are not returned. This damping may be expressed in more usual terms, as an acoustic damping ratio, by writing

$$\zeta_a = \frac{1}{2n\pi}\log\left(\frac{1+\alpha}{1-\alpha}\right) \approx \frac{\alpha}{n\pi} \quad (3)$$

The second, approximate, form is found by expanding the log term in a Taylor series. The approximate form is within 3% for acoustic damping ratios up to 0.1

The pressure at the piston for just one mode is given by

$$P = \frac{i\rho c^2}{SL} \frac{(-1)^n Q + Q_p}{\frac{nc\pi}{L} - \omega + \frac{ic\alpha}{L}} \quad (4)$$

Here the denominator has the familiar form of a resonant pole with a natural frequency $\omega = n\pi c / L$. It should be stressed that the natural frequencies and damping ratios are those for the uncoupled acoustic system. In a finite element analysis they would be found directly in the modal form.

The mechanical system is modelled in the usual form by

$$\sigma = \frac{F - PS}{\omega_s^2 - \omega^2 + 2i\zeta_s\omega\omega_s} \quad (5)$$

where σ is the force in the spring, F an externally applied force, P the pressure from the acoustic loading (as given by Equation 1), ω_s the structural natural frequency in the absence of acoustic loading and ζ_s the structural damping ratio. Here the force in the spring has been determined. This reflects the usual acoustic-structural problem where stresses must be calculated. As with the acoustic system a simplification may be introduced by expanding about the resonance frequency and hence reducing the order of the expression. This gives the force in the spring as

$$\sigma = \frac{F - PS}{2(\omega_s - \omega + i\zeta_s\omega_s)} \quad (6)$$

In a more general analysis this equation would have been obtained by extracting one of the vibration modes from a finite element analysis of the structural system in which the fluid is absent. The relevant mode would have been one which has a similar natural frequency to an acoustic frequency.

Finally the two equations describing the acoustic and structural systems may be combined to give

$$\sigma = \frac{Q c \rho}{2 n \pi} \frac{i(-1)^n}{\left(1 - \frac{\omega}{\omega_s} + i \zeta_s\right)\left(1 - \frac{\omega L}{c n \pi} + i \zeta_a\right) - \frac{\omega L}{c n \pi}\beta} \tag{7}$$

where the parameter β has been introduced to collect various terms of the original equations. The parameter β is given by

$$\beta = \frac{\rho c^2 S}{2 k L} \tag{8}$$

where k is the stiffness of the spring connected to the piston. The parameter β is the coupling parameter and is equal to half the ratio of the compressibility of the fluid to the stiffness of the structural spring. This term also includes the area of the piston that in a more general analysis would be a composite mixture of acoustic and structural mode shapes. (For a similar formulation, for this type of problem, involving mode shapes see Fahy (1985).)

Equation 7 is the general coupled equation for an acoustic and structural oscillator. This equation will be explored in the remainder of this paper. Note that we have the following five parameters on which the problem depends: ω_s, $(c\, n\, \pi / L)$, ζ_s, ζ_a and β which are respectively the uncoupled structural natural frequency, the uncoupled acoustic natural frequencies, the uncoupled structural damping ratio, the uncoupled acoustic natural frequency, and the coupling parameter.

3. EXAMPLES OF COUPLED SYSTEM

We wish to find when Equation 7 gives a maximum response. Figures 2 and 3 give examples of two types of frequency response functions for the spring force modelled by Equation 7. In both figures the frequency axis is normalised on the acoustic natural frequency of the uncoupled system and the force in the spring is normalised by dividing by $\rho\, c\, Q$. This results in the frequency axis being 1.0 at the frequency corresponding to the uncoupled acoustic natural frequency.

The difference between Figure 2 and Figure 3 is the relationship between the coupling parameter β and the acoustic and structural damping ratios. In Figure 1 $\beta > \zeta_s \zeta_a$ while in Figure 2 $\beta < \zeta_s \zeta_a$. This criterion emerges from the analysis as the major factor in controlling the type of frequency response functions found. Within each figure the coupling parameter and the damping ratios have been kept constant and the various curves correspond to different values of uncoupled structural natural frequency. Such different values could be obtained by varying the mass of the piston arrangement.

In Figure 2 each frequency response function has two resonance peaks close to the uncoupled acoustic natural frequency. This is perhaps not surprising since there are two resonant systems. Also, note that the largest peak does not occur in the case when the uncoupled acoustic and structural frequencies are equal but when the structural frequency is less than the acoustic frequency. The peak with the maximum possible value, for the parameters used to construct the figure, is shown with an unbroken line.

In Figure 3 the coupling parameter and the damping ratios are again kept constant and each curve corresponds to a different value of uncoupled structural frequency. In this case the largest resonant peak, shown with the unbroken line, is a single peak and is located at the coincidence of the uncoupled acoustic and structural parameters. The other frequency response functions, all of which have resonance peaks less than the curve with the maximum peak, again have two peaks although further exploration reveals that this is not necessarily always true. However, it is true that for $\beta < \zeta_s \zeta_a$ the largest peak is always associated with a frequency response function with a single peak.

Finally it should be noted that if a simulation of the coupled system had been undertaken then just one of the curves in Figure 2 or Figure 3 would have been calculated and the height of the largest peak would have been taken as the worst case. However, the actual system when built would be different to that simulated and may correspond to the largest peak in the Figures. Hence the objective of finding the height of the largest peak.

4. ANALYSIS OF COUPLED SYSTEM

The mathematics required to determine the global maximum of Figure 2 or Figure 3 from Equation 7 requires more space than is allowed in this paper.

Consequently, the method will be described and illustrated with the mathematical details left to a companion paper. Essentially the method used is that of the calculus of variations. The starting point is to rewrite Equation 7 in terms of new variables as follows

$$\sigma = \frac{Q c \rho}{2n\pi} \frac{-i}{(1-\Omega_s + i\,\zeta_s)(1-\Omega_a + i\,\zeta_a) - \Omega_a \beta} \tag{9}$$

where there are now two frequency variables Ω_s and Ω_a and for simplicity we examine the case where n = 1. The two frequency variables are given by

$$\Omega_s = \frac{\omega}{\omega_s} \text{ and } \Omega_a = \frac{\omega L}{c\pi} \tag{10}$$

making the problem one involving frequency in two dimensions.

The maximum value of Equation 9 may now be attempted by treating it as a two dimensional maximisation problem. A typical contour plot of Equation 9 is given by Figure 4. This plot corresponds to the case in Figure 2 where each frequency response function has two peaks. The global maximum peak is located at point *A* with the second peak at point *B*. The point *C* is a saddle point between the two main peaks and it can be shown that *A*, *B* and *C* lie on a straight line. The curved lines, in a hyperbolic form labelled $\mathbf{G_1}$ and $\mathbf{G_2}$, are the locations of the natural frequencies of the un-damped system. Note that the peaks are located close to, but not on these lines.

In Figure 4 the frequency response functions illustrated in Figure 2 are found along straight sectional lines drawn from the origin. Thus sections of the two-dimensional function along lines *P, Q, R, S* and *T* correspond to the five frequency response functions in Figure 2. Line *P*, corresponding to ω_s / ω_a = 0.7, does not go through either of the peaks but does see maxima where it crosses the main ridges corresponding to the natural frequencies. Line *Q*, corresponding to ω_s / ω_a = 0.849 , goes through the maximum peak at *A* while line *S* corresponding to ω_s / ω_a = 1.11, goes through the second global peak at *B*. Line *R*, corresponding to ω_s / ω_a = 0.988, goes through the saddle point at *C* while section *T* corresponds to the section having ω_s / ω_a = 1.2.

The formal approach to finding expressions for the global maxima would be to take the absolute value of the denominator of Equation 9 and then to take derivatives with respect to Ω_s and Ω_a The two resulting equations could then be solved simultaneously to find the locations of the minima of the denominator. Finally the location of the minima could be substituted back into Equation 9 to find expressions for the maxima. This has been attempted using a computer algebra system (Mathematica 2007) but yielded equations that were too complex to be of much use. Instead the computer algebra system was used to develop solutions using the calculus of variations and then to simplify them using Taylor series.

The results of the analysis gave rise to the following expressions for the maxima of Equation 9.

For the case where there are two peaks, the criteria for which is

$$\beta > \zeta_s \zeta_a$$

The global peak is given by

$$\sigma = \frac{\rho c Q}{2\pi n \sqrt{\beta}\left(2\sqrt{\zeta_s \zeta_a} - \zeta_a \sqrt{\beta - \zeta_s \zeta_a}\right)} \tag{11}$$

and the second peak is given by

$$\sigma = \frac{\rho c Q}{2\pi n \sqrt{\beta}\left(2\sqrt{\zeta_s \zeta_a} + \zeta_a \sqrt{\beta - \zeta_s \zeta_a}\right)} \tag{12.}$$

For the case where there is one peak for which

$$\beta < \zeta_s \zeta_a$$

The peak is given by

$$\sigma = \frac{\rho c Q}{2\pi n (\beta + \zeta_s \zeta_a)} \tag{13}$$

All the above equations have been checked using numerical values and, for example, agree with the peak heights illustrated in Figures 2 and 3.

5. DISCUSSION

Three approximations have been made in developing the above formula. The validity of these approximations is as follows.

5.1 Approximation 1

It has been assumed that the combined resonance of the system can be deduced by considering the interaction of one structural mode and one acoustic mode and ignoring other structural and acoustic modes. It would be nice to include several acoustic and structural modes all interacting but this would involve three or more close frequencies. Such an analysis may be attempted now that some of the features of two close natural frequencies have been deduced. The key point to start such an analysis is to determine a criterion for when two modes are close. This may be deducible from the present analysis but remains to be attempted. For the moment the above analysis is only valid if there is no close acoustic or structural natural frequency in addition to those being considered.

5.2 Approximation 2

The acoustic and structural modes have been simplified by considering only the response due to the poles in the positive frequency half plane. The errors due to this approximation have been studied and are much less than those due to possible neighbouring modes that have been ignored. Thus if it is valid to make approximation 1 then it is certainly valid to make this approximation.

5.3 Approximation 3

In developing the formula for the maximum response use has been made of Taylor series. The series expansion is for small acoustic damping values. The errors involved are being investigated but appear small and thus the formula should be valid for the typical case of small acoustic damping ratio.

6. CONCLUSIONS

The following conclusions may be drawn.

1. The acoustic and structural systems couple to produce one or two resonant modes.
2. The maximum response may not occur when the uncoupled acoustic and structural resonant frequencies are equal but at two frequencies slightly less than and slightly more than either uncoupled frequency.
3. Equation 11 gives the maximum possible response of the coupled system. It involves coupling parameter β and structural and acoustic damping ratios. The coupling parameter β depends on the ratio of the bulk stiffness of the fluid and the structural stiffness of the structure. It also depends on the area over which the fluid and structure are coupled.
4. The Equation for the maximum response of the coupled system can be used in design work to combine analyses of acoustic and structural systems that have been studied separately.

7. ACKNOWLEDGEMENTS

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8. REFERENCES

Fahy FJ, 1985, Sound and Structural Vibration: Radiation, Transmission and Response, Academic Press Inc. (London) Ltd. ISBN 0-12-247670-0.

Pippard AB, 1989, The physics of vibration. Cambridge University Press. ISBN 0521 37200

Mathematica 2007 Wolfram Research, Champaign IL

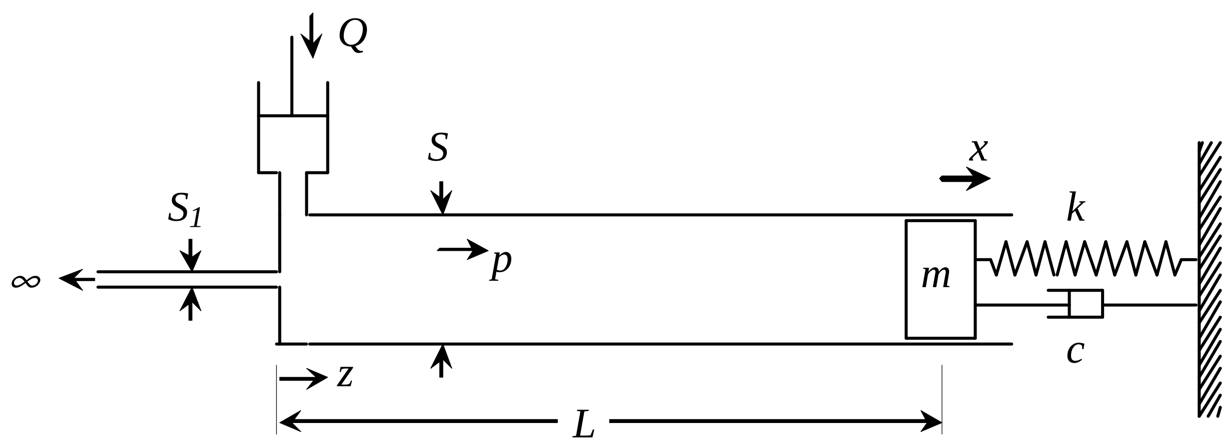

Figure 1. A simple system for developing the analysis of coupled acoustic-structural systems. Acoustic waves are set up in the cavity of length, *L*, and with cross-sectional area, *S*, by the acoustic source of volume velocity, *Q*, on the left. The piston arrangement on the right is a mass-spring-damper that acts as a resonant structural system. The small pipe on the left with cross-sectional area, S_1, allows waves to pass out of the system and thus provides acoustic damping for the cavity.

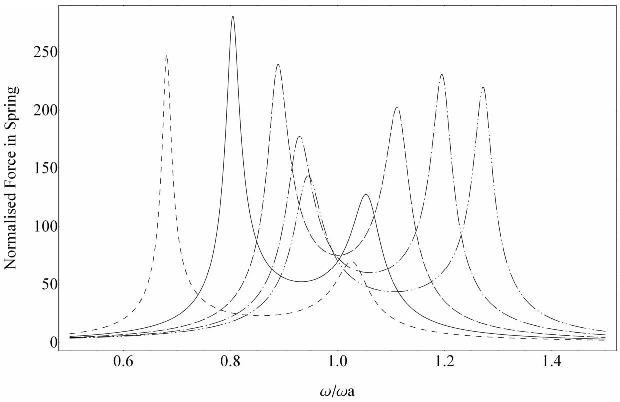

Figure 2. Frequency response functions for coupled system with ζ_s = 0.01, ζ_a = 0.03, β = 0.013 and ω_s/ω_a = ---- 0.7, ——— 0.849, - - - 0.988, -·-·- 1.11,

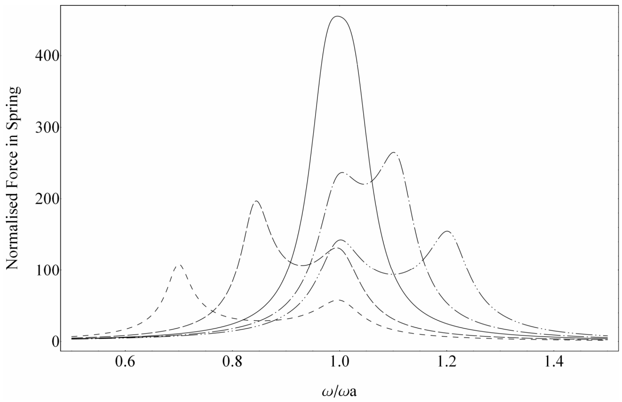

Figure 3. Frequency response functions for coupled system with ζ_s = 0.03, ζ_a = 0.04, β = 0.001 and ω_s/ω_a = ---- 0.7, - - - 0.845, ——— 1.0, -·-·- 1.1,

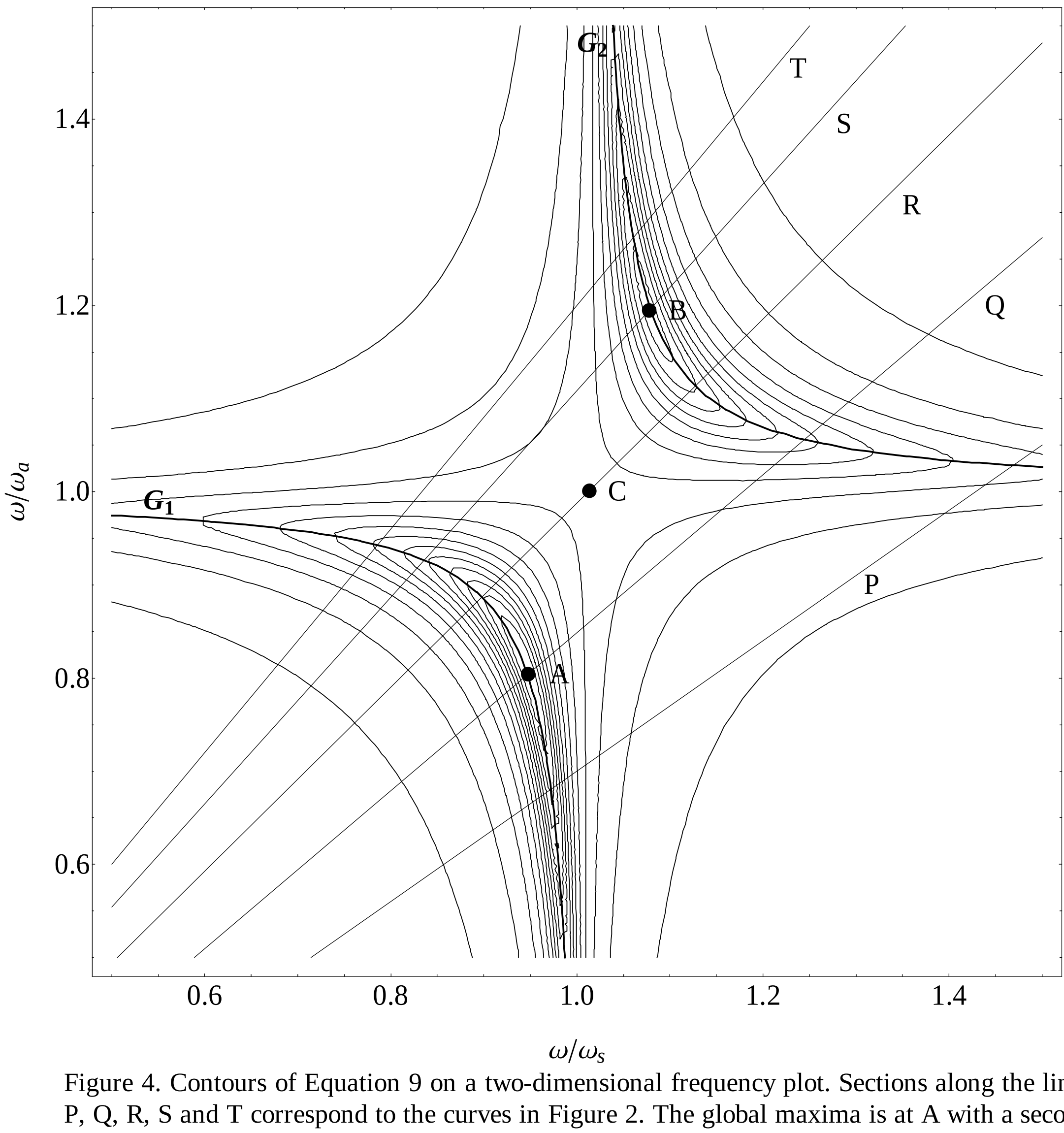

Figure 4. Contours of Equation 9 on a two-dimensional frequency plot. Sections along the lines P, Q, R, S and T correspond to the curves in Figure 2. The global maxima is at A with a second peak at B and a saddle at C. Lines G_1 and G_2 correspond to the locus of natural frequencies of the undamped coupled system.