# NONLINEAR STABILITY OF A FLUID-CONVEYING CANTILEVERED PIPE WITH END MASS IN CASE OF HORIZONTAL EXCITATION AT THE UPPER END 

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#### Abstract

Nonplanar vibrations of a cantilevered pipe with an end mass is studied. We have already clarified the nonplanar vibrations with a single frequency component when the pipe conveys fluid whose velocity is slightly over the critical value, above which the lateral vibration of the pipe is self-excited due to the internal flow. Moreover, for the case that the upper end of the pipe is excited periodically in the horizontal direction, we have shown in the previous paper that the nonplanar limit cycle motions start complex spatial transients and settle down to stationary planar forced-excited vibration when the excitation frequency is near the nonplanar limit cycle frequency. The purpose of this paper is to examine the stability of the nonplanar pipe vibrations when the nonplanar self-excited pipe vibrations are subjected to the excitation at the upper end. A set of ordinary differential equations, which govern the amplitudes and phases of unstable mode vibration and contain the effect of excitation at the upper end are derived. Stability analysis of these equations clarifies the nonlinear interactions between nonplanar self-excited pipe vibrations and the forced excitation. Second, the experiments are conducted with a silicon rubber pipe conveying water, confirming the dynamic features of pipe vibrations for the horizontal excitation.


## 1 INTRODUCTION

There is a long history of studies of the dynamics of a fluid conveying pipe, which are written in Paidoussis(1998) [1].Flow-
induced nonplanar vibration of a fluid-conveying pipe, which has been studied theoretically by Bajaj and Sethna (1984) [2], Troger and Steindl (1991) et al. [3], is one of the attractive phenomena from the viewpoint of nonlinear dynamics (Paidoussis and Li, 1993) [4]. Bajaj and Sethna (1991) also studied theoretically and experimentally three-dimensional oscillatory motions of a cantilevered pipe, where small different bending stiffnesses in two mutually perpendicular directions are imposed to break the rotational symmetry [5]. Furthermore, Copeland and Moon (1992) clarified experimentally that the addition of an end mass to a cantilevered pipe yields a rotational symmetric system with many types of nonplanar vibrations [6]. Yoshizawa et al.(1998) examined theoretically and experimentally the effects of the end mass on the nonplanar pipe vibration [7]. Moreover, Wadham-Gagnon et al.(2007) derived the equations of motion using Hamilton's principle [8]. The equations of motion are derived from Newton's second law in this study.

In this paper, nonlinear lateral vibrations of a cantilevered pipe, which is hung vertically with an end mass and conveys fluid, are examined for the case that the upper end of the pipe is excited periodically in a horizontal plane. The fluid velocity is slightly over the critical value, above which lateral pipe vibration is self-excited due to an internal flow.

First, the four first-order ordinary differential equations governing the amplitudes and phases of $v$ and $w$ are derived from the nonlinear partial differential equations of nonplanar pipe vibration, where $v$ is the lateral deflection of the pipe in the direction


FIGURE 1. Analytical model
of the forced excitation and $w$ is the lateral deflection of the pipe perpendicular to $v$. Those equations are coupled through the nonlinear terms which are caused by cubic nonlinearities of $v$ and $w$ in the equations of nonplanar pipe vibration. The interactions between the forced and flow-induced pipe vibrations are examined theoretically by solving numerically the obtained equations of the amplitudes and phases.

Second, the experiments were conducted with a silicon rubber pipe conveying water. The spatial behaviors of the fluidconveying pipe were observed in cases of non-forced and forced excitations at the upper end, using the image processing system that is based on the images from two CCD cameras. As a result, the typical effect of horizontal excitation on nonplanar flowinduced pipe vibration, predicted in the theory, was confirmed qualitatively by experiment.

## 2 BASIC EQUATIONS

### 2.1 EQUATIONS OF MOTION

The system under consideration (Fig.1), consists of a flexible pipe with an end mass $M$, conveying an incompressible fluid, which is discharged into an atmosphere at the free end of the pipe.

The pipe of length $l$, flexural rigidity $E I$, mass per unit length m and cross-sectional flow area $S$, is hung vertically under the influence of gravity $\boldsymbol{g}$ in its equilibrium state. The pipe is sufficiently long compared with its radius, and its centerline is inex-
tensible. The internal fluid of density $\rho$ is incompressible. The axial fluid velocity $V_{s}$ relative to the pipe motion is assumed to be maintained at constant under the condition that the frictional force between the fluid and the pipe wall is large (Bajaj, Sethna and Lundgren, 1980) [9].

We use two systems of co-ordinates : a fixed system $X-Y-$ $Z$, and a moving system $x-y-z$, to describe the motion of a pipe. The origin of the moving system is taken to coincide with the upper clamped end of the pipe, which is excited periodically in a horizontal direction as follows:

$$
\begin{equation*}
Y_{0}=\delta Y \sin N t \tag{1}
\end{equation*}
$$

where $\delta Y$ and $N$ are the amplitude of excitation and the frequency of excitation, respectively. Let $v(s, t)$ and $w(s, t)$ be the deflections of the pipe centerline in the $y$ and $z$ directions respectively, which are expressed as functions of co-ordinate $s$ along the pipe axis and time $t$.

Then the equations governing the spatial behavior of the pipe are derived under the assumptions that $v$ and $w$ are small but finite, and the pipe has no torsion about its centerline (Watanabe, 1996) [10].

Introducing the dimensionless variables which carry the asterisk, i.e. $v=l v^{*}, w=l w^{*}, s=l s^{*}, t=\sqrt{(m+\rho S) l^{4} /(E I)} t^{*}$, and retaining terms up to the third order of $v^{*}$ and $w^{*}$, the dimensionless equation of the pipe motion in the xy plane is expressed as follows:

$$
\begin{aligned}
& \ddot{v}+v^{\prime \prime \prime \prime}-\gamma\left[(\alpha+1-s) v^{\prime}\right]^{\prime}+V_{s}^{2} v^{\prime \prime}+2 \sqrt{\beta} V_{s} \dot{v}^{\prime} \\
& =-v^{\prime \prime} \int_{s}^{1}\left[-\frac{1}{2} \frac{\partial^{2}}{\partial t^{2}}\left\{\int_{0}^{s}\left(v^{2}+w^{2}\right) d s\right\}+v^{\prime} \ddot{v}+w^{\prime} \ddot{w}\right] d s \\
& -\frac{1}{2}\left[v^{\prime} \frac{\partial^{2}}{\partial t^{2}}\left\{\int_{0}^{s}\left(v^{\prime 2}+w^{2}\right) d s\right\}-v^{\prime 2} \ddot{v}\right] \\
& -\frac{1}{2} \ddot{w}\left\{v^{\prime} w^{\prime}-\int_{0}^{s}\left(v^{\prime} w^{\prime \prime}-v^{\prime \prime} w^{\prime}\right) d s\right\} \\
& +v^{\prime \prime} \int_{s}^{1}\left(v^{\prime \prime} v^{\prime \prime \prime}+w^{\prime \prime} w^{\prime \prime \prime}\right) d s \\
& -\frac{1}{2}\left[v^{\prime}\left(v^{\prime} v^{\prime \prime}+w^{\prime} w^{\prime \prime}\right)+w^{\prime \prime} \int_{0}^{s}\left(v^{\prime} w^{\prime \prime}-v^{\prime \prime} w^{\prime}\right) d s\right]^{\prime \prime} \\
& -\frac{1}{2} \gamma v^{\prime \prime} \int_{s}^{1}\left(v^{\prime 2}+w^{2}\right) d s-\frac{1}{2} \gamma w^{\prime} \int_{0}^{s}\left(v^{\prime} w^{\prime \prime}-v^{\prime \prime} w^{\prime}\right) d s \\
& +\frac{1}{2}(1-s) \gamma\left[v^{\prime}\left(v^{\prime} v^{\prime \prime}+w^{\prime} w^{\prime \prime}\right)+w^{\prime \prime} \int_{0}^{s}\left(v^{\prime} w^{\prime \prime}-v^{\prime \prime} w^{\prime}\right) d s\right] \\
& +\alpha v^{\prime \prime}\left[\frac{1}{2} \frac{\partial^{2}}{\partial t^{2}}\left\{\int_{0}^{1}\left(v^{2}+w^{2}\right) d s\right\}-v_{1}^{\prime} \ddot{v}_{1}-w_{1}^{\prime} \ddot{w}_{1}\right]
\end{aligned}
$$

$$
\begin{align*}
& +\frac{1}{2} \alpha \gamma\left[v^{\prime}\left(v^{\prime} v^{\prime \prime}+w^{\prime} w^{\prime \prime}\right)+w^{\prime \prime} \int_{0}^{s}\left(v^{\prime} w^{\prime \prime}-v^{\prime \prime} w^{\prime}\right) d s-v^{\prime \prime}\left(v_{1}^{\prime 2}+w_{1}^{\prime 2}\right)\right] \\
& -\frac{1}{2} V_{s}^{2}\left[v^{\prime}\left(v^{\prime} v^{\prime \prime}+w^{\prime} w^{\prime \prime}\right)+w^{\prime \prime} \int_{0}^{s}\left(v^{\prime} w^{\prime \prime}-v^{\prime \prime} w^{\prime}\right) d s\right] \\
& -\sqrt{\beta} V_{s}\left[v^{\prime}\left(v^{\prime} \dot{v}^{\prime}+w^{\prime} \dot{w}^{\prime}\right)+\dot{w}^{\prime} \int_{0}^{s}\left(v^{\prime} w^{\prime \prime}-v^{\prime \prime} w^{\prime}\right) d s\right] \\
& +k v^{2} \sin v t \tag{2}
\end{align*}
$$

where ' and ' denote the derivatives with respect to $t$ and $s$, respectively. The asterisks indicating the dimensionless variables are omitted in equation (2) and henceforward.

The boundary conditions for both ends of the pipe in xy plane are expressed as follows:

$$
\left\{\begin{aligned}
s=0: & v=v^{\prime}=0 \\
s=1: & v^{\prime \prime}=0 \\
& v^{\prime \prime \prime}-\alpha\left(\gamma v^{\prime}+\ddot{v}\right) \\
& =\frac{1}{2} \alpha\left[v^{\prime} \frac{\partial^{2}}{\partial t^{2}}\left\{\int_{0}^{s}\left(v^{2}+w^{2}\right) d s\right\}\right. \\
& \left.-\ddot{v} v^{\prime 2}-\ddot{w}\left\{v^{\prime} w^{\prime}-\int_{0}^{1}\left(v^{\prime} w^{\prime \prime}-v^{\prime \prime} w^{\prime}\right) d s\right\}\right] \\
& -\frac{1}{2}\left[v^{\prime}\left(v^{\prime} v^{\prime \prime}+w^{\prime} w^{\prime \prime}\right)+w^{\prime \prime} \int_{0}^{s}\left(v^{\prime} w^{\prime \prime}-v^{\prime \prime} w^{\prime}\right) d s\right]^{\prime} \\
& +\alpha k v^{2} \sin v t
\end{aligned}\right.
$$

The equation of the pipe motion in the z -x plane and its boundary conditions, are expressed by exchanging $v$ for $w$ and $w$ for $v$, and taking off the excitation terms proportional to $k v^{2} \sin v t$ in equations (2) and (3). As a result, the spatial behavior of the pipe is described by two equations and eight boundary conditions with respect to $v$ and $w$.

There are six dimensionless parameters involved in equations (2), (3) and their symmetric equations, i.e. the dimensionless velocity $V_{s}=v_{s} / \sqrt{E I /\left(\rho S l^{2}\right)}$, the ratio of the lumped mass to the total mass $\alpha=M /(m+\rho S) l$, the ratio of the fluid mass to the total mass $\beta=\rho S /(m+\rho S)$, the ratio of the gravity force to the elastic force of the pipe $\gamma=(m+\rho S) g l^{3} / E I$, the dimensionless amplitude of excitation $k=\delta Y / l,(k \ll 1)$ and the dimensionless frequency of excitation $v=N \sqrt{(m+\rho S) l^{4} /(E I)}$.

### 2.2 Equation in Vector Form

By defining

$$
\boldsymbol{v}_{v}={ }^{t}\left[\begin{array}{ll}
v & \partial v / \partial t
\end{array}\right], \quad \boldsymbol{v}_{w}={ }^{t}\left[\begin{array}{ll}
w & \partial w / \partial t \tag{4}
\end{array}\right]
$$

the governing equations of $\boldsymbol{v}_{i}(i=v, w)$ are expressed in the vector form as follows:

$$
\begin{gather*}
\frac{\partial \boldsymbol{v}_{i}}{\partial t}=L \boldsymbol{v}_{i}+\boldsymbol{N}_{i},  \tag{5}\\
\left\{\begin{array}{l}
s=0: B_{1} \boldsymbol{v}_{i}=\mathbf{0} \\
s=1: B_{2} \boldsymbol{v}_{i}=B_{3} \boldsymbol{v}_{i}-\boldsymbol{N}_{b i}
\end{array}\right. \tag{6}
\end{gather*}
$$

from two equations and eight boundary conditions with $v$ and $w$, where

$$
\begin{aligned}
L & =\left[\begin{array}{cc}
0 & 1 \\
L_{21} & -2 \sqrt{\beta} V_{s}(\cdot)^{\prime}
\end{array}\right] \\
\boldsymbol{N}_{v} & =\left[\begin{array}{c}
0 \\
n(v, w)+k v^{2} \sin v t
\end{array}\right], \quad \boldsymbol{N}_{w}=\left[\begin{array}{c}
0 \\
n(v, w)
\end{array}\right] \\
B_{1} & =\left[\begin{array}{cc}
1 & 0 \\
(\cdot)^{\prime} & 0
\end{array}\right], \quad B_{2}=\left[\begin{array}{ll}
0 & 0 \\
0 & \alpha
\end{array}\right] \\
B_{3} & =\left[\begin{array}{cc}
(\cdot \cdot)^{\prime \prime} & 0 \\
(\cdot)^{\prime \prime \prime}-\alpha \gamma(\cdot)^{\prime} & 0
\end{array}\right] \\
\boldsymbol{N}_{b v} & =\left[\begin{array}{cc}
0 \\
b(w, v)+\alpha k v^{2} \sin v t
\end{array}\right] \\
\boldsymbol{N}_{b w} & =\left[\begin{array}{c}
0 \\
b(w, v)
\end{array}\right]
\end{aligned}
$$

and

$$
L_{21}=-(\cdot)^{\prime \prime \prime \prime}+\gamma\left[(\alpha+1-s)(\cdot)^{\prime}\right]^{\prime}-V_{s}^{2}(\cdot)^{\prime \prime}
$$

$n(v, w), n(w, v), b(v, w)$ and $b(w, v)$ in Eqs. (5) and (6) are expressed as the third order nonlinear polynomials with respect to $v$ and $w$ and $(\cdot)^{\prime}$ stands for $\partial v / \partial s$.

## 3 LINEAR STABILITY

Neglecting the nonlinear terms with respect to $v, w$, and putting $k=0$ in Eqs. (5) and (6), $\boldsymbol{v}_{v}$ and $\boldsymbol{v}_{w}$ become independent of each other, and are described by the same equations and boundary conditions. Therefore they have the identical linear eigenvalues and eigenfunctions. We solve the boundary value problem for $\boldsymbol{v}_{v}$ in this section. Letting $\boldsymbol{v}_{v}=\boldsymbol{q}_{v} e^{\lambda_{v} t}, \boldsymbol{q}_{v}={ }^{t}\left[\Phi_{v 1}(s), \Phi_{v 2}(s)\right]$ and substituting them into Eqs. (5) and (6), we can cast into the eigenvalue problem

$$
\begin{equation*}
\lambda_{v} \boldsymbol{q}_{v}=L \boldsymbol{q}_{v} \tag{7}
\end{equation*}
$$

$$
\begin{equation*}
s=0: B_{1} \boldsymbol{q}_{v}=\mathbf{0}, \quad s=1: B_{4} \boldsymbol{q}_{v}=\mathbf{0} \tag{8}
\end{equation*}
$$

where

$$
B_{4}=\left[\begin{array}{cc}
(\cdot)^{\prime \prime} & 0 \\
(\cdot)^{\prime \prime \prime}-\alpha \gamma & -\alpha \lambda_{v}
\end{array}\right]
$$

The eigenvalue $\lambda_{v}$, being the root of the complex characteristic equation which is symbolically described by

$$
\begin{equation*}
f\left(\lambda_{v}: V_{s}, \alpha, \beta, \gamma\right)=0 \tag{9}
\end{equation*}
$$

can be found numerically from the condition that Eqs. (7) and (8) have a non-trivial solution $\boldsymbol{q}_{\boldsymbol{v}}$. The eigenvalue $\lambda_{v}$ is equal to $i\left(\omega_{r}+i \omega_{i}\right)$, where $\omega_{r}$ is the linear natural frequency and $\omega_{i}$ corresponds to the damping coefficient.

Figure 2 shows the complex natural frequencies $\omega=\omega_{r}+i \omega_{i}$ as a function of $V_{s}$ for the first three modes of the system in the case of $\alpha=0.13, \beta=0.26$ and $\gamma=20.6$. Those values of $\alpha, \beta$ and $\gamma$, which are used in the numerical examples henceforward, are equal to the values of the experimental ones in Section 5. When $\omega_{i}$ is negative, the pipe is self-excited.

The lowest value of $V_{s}$, at which the flow-induced pipe vibration appears for the second mode, is 6.03 and will be reffered to as the critical flow velocity $V_{c r}$. The natural frequency $\omega_{r}$ of the second mode is 16.1 at $V_{s}=6.03$. In the next section, we derive the nonlinear first-order ordinary differential equations governing the amplitudes and phases of $v$ and $w$ for the case of forced excitation, under the condition that $V_{s}$ is slightly above $V_{c r}$.

An eigenvector $\boldsymbol{q}_{v}$ of the second mode, which is used in the following section, can be found in the form of a power series of $s$, and satisfies the condition $\left\langle\boldsymbol{q}_{v}, \boldsymbol{q}_{v}\right\rangle=1$ where brackets denote the inner product $\langle\boldsymbol{x}, \boldsymbol{y}\rangle=\int_{0}^{1} t \boldsymbol{x}(s) \overline{\boldsymbol{y}(s)} d s$.

Moreover from the condition:

$$
\begin{equation*}
\left\langle L \boldsymbol{q}_{v}, \boldsymbol{q}_{v}^{*}\right\rangle=\left\langle\boldsymbol{q}_{v}, L^{*} \boldsymbol{q}_{v}^{*}\right\rangle \tag{10}
\end{equation*}
$$

we get the equations of the adjoint vector $\boldsymbol{q}_{v}^{*}={ }^{t}\left[\Psi_{11}(s), \Psi_{12}(s)\right]$ of $\boldsymbol{q}_{v}$ as follows:

$$
\begin{equation*}
\overline{\lambda_{v}} \boldsymbol{q}_{v}^{*}=L^{*} \boldsymbol{q}_{v}^{*} \tag{11}
\end{equation*}
$$

$$
\begin{equation*}
s=0: C_{1} \boldsymbol{q}_{v}^{*}=\mathbf{0}, \quad s=1: C_{2} \boldsymbol{q}_{v}^{*}=\mathbf{0} \tag{12}
\end{equation*}
$$

where

$$
\begin{aligned}
L^{*} & =\left[\begin{array}{lc}
0 & L_{12}^{*} \\
1 & 2 \sqrt{\beta} V_{s}(\cdot)^{\prime}
\end{array}\right] \\
C_{1} & =\left[\begin{array}{cc}
0 & 1 \\
0 & (\cdot)^{\prime}
\end{array}\right], \quad C_{2}=\left[\begin{array}{cc}
0 & -(\cdot)^{\prime \prime}-V_{s}^{2} \\
0 & C_{22}
\end{array}\right]
\end{aligned}
$$



FIGURE 2. The complex natural frequencies $\omega=\omega_{r}+i \omega_{i}$ as a function of $V_{S}$ for the first three modes $(\alpha=0.13, \beta=0.26$ and $\gamma=20.6)$
and

$$
\begin{aligned}
& L_{12}^{*}=-(\cdot)^{\prime \prime \prime \prime}+\gamma\left\{(\alpha+1-s)(\cdot)^{\prime}\right\}^{\prime}-V_{s}^{2}(\cdot)^{\prime \prime} \\
& C_{22}=(\cdot)^{\prime \prime \prime}+\left(V_{s}^{2}-\alpha \gamma\right)(\cdot)^{\prime}-2 \overline{\lambda_{v}} \sqrt{\beta} V_{s}-\alpha{\overline{\lambda_{v}}}^{2}
\end{aligned}
$$

The adjoint vector $\boldsymbol{q}_{v}^{*}$, which is expressed in the form of a power series of $s$, also satisfies the condition $\left\langle\boldsymbol{q}_{v}, \boldsymbol{q}_{v}^{*}\right\rangle=1$.

## 4 NONLINEAR STABILITY

### 4.1 Amplitude Equation

The equations governing the amplitudes and phases of $\boldsymbol{v}$ and $\boldsymbol{w}$ are derived for the case when the flow velocity $V_{s}$ is near the critical velocity $V_{c r}$.

The Banach space, which includes $\boldsymbol{v}_{v}$ and $\boldsymbol{v}_{w}$, is expressed as $\boldsymbol{Z}=\boldsymbol{X} \oplus \boldsymbol{M}$ (Carr,1981; Paidoussis and Li,1993) [11]. $\boldsymbol{X}$ is the eigenspace spanned by the eigenvectors $\boldsymbol{q}_{v}$ and $\boldsymbol{q}_{w}$, which correspond to the linear unstable vibration modes of $\boldsymbol{v}_{v}$ and $\boldsymbol{v}_{w}$, respectively. $\boldsymbol{M}$ is the subspace of $\boldsymbol{X}$. Therefore, $\boldsymbol{v}_{i}(i=v, w)$ are expressed as follows:

$$
\begin{equation*}
\boldsymbol{v}_{i}(s, t)=a_{i}(t) \boldsymbol{q}_{i}(s)+\boldsymbol{y}_{i}(s, t)+C . C ., \tag{13}
\end{equation*}
$$

where $\boldsymbol{y}_{\boldsymbol{v}}$ and $\boldsymbol{y}_{\boldsymbol{w}}$ are the elements of $\boldsymbol{M}$.
Using the projection $P_{i}$ onto $\boldsymbol{X}$, Eq. (5) with boundary conditions (6) are decomposed as follows:

$$
\begin{equation*}
P_{i} \frac{\partial \boldsymbol{v}_{i}}{\partial t}=P_{i} L \boldsymbol{v}_{i}+P_{i} \boldsymbol{N}_{i}, \quad(i=v, w) \tag{14}
\end{equation*}
$$

where $P_{i} \boldsymbol{x}=\left\langle\boldsymbol{x}, \boldsymbol{q}_{i}^{*}\right\rangle \boldsymbol{q}_{i}$.

From equations (14), the equations of $a_{i}$ are derived as follows:

$$
\begin{equation*}
\dot{a}_{i}=\lambda_{i} a_{i}+\left(\xi_{1} a_{i}^{3}+\xi_{2} a_{i} a_{j}^{2}+\xi_{3}\left|a_{i}\right|^{2} a_{i}+\xi_{4}\left|a_{j}\right|^{2} a_{i}+\xi_{5} \bar{a}_{i} a_{j}^{2}\right)+f_{i} \tag{15}
\end{equation*}
$$

where $j=w, f_{v}=\xi_{6} k v^{2} \sin v t$ for $i=v$, and $j=v, f_{w}=0$ for $i=w$. The constant coefficients $\xi_{1}, \xi_{2}, \cdots, \xi_{6}$ in Eqs. (15), are numerically determined as functions of $\alpha, \beta, \gamma$ and $V_{s}$.

Letting $a_{v}=h_{v} e^{i\left(\omega_{r} t+\phi\right)} / 2$ and $a_{w}=h_{w} e^{i\left(\omega_{r} t+\psi\right)} / 2$, separating the real and imaginary parts of Eqs. (15), and averaging them by the period $2 \pi / \omega_{r}$ under the following assumption:

$$
\begin{equation*}
v \equiv \omega_{r}+\sigma, \quad\left(|\sigma| \ll \omega_{r}\right), \tag{16}
\end{equation*}
$$

we obtain the nonlinear first-order ordinary differential equations which govern the amplitude $h_{v}$ and the phase $\phi$ of $a_{v}$, and the amplitude $h_{w}$ and the phase $\psi$ of $a_{w}$. Here, $\omega_{r}$ is the linear natural frequency of the second mode of the lateral pipe vibration.

Furthermore, we define the phase difference $\Omega$ between $a_{w}$ and $a_{v}$, and the phase difference $\eta$ between $Y_{0}$ and $a_{v}$, respectively, as follows:

$$
\begin{equation*}
\Omega=\psi-\phi, \quad \eta=\sigma t-\phi \tag{17}
\end{equation*}
$$

The equations of $\Omega$ and $\eta$ are obtained from the equations of $\phi$ and $\psi$. Finally the autonomous equations governing $h_{v}, \Omega, h_{w}$ and $\eta$ are expressed as follows:

$$
\begin{align*}
\dot{h_{v}} & =-\omega_{i} h_{v}+\left(\xi_{4 r e}+\xi_{5 r e}\right) h_{v}^{3} / 4 \\
& +\left(\xi_{4 r e}+\xi_{5 r e} \cos 2 \Omega-\xi_{5 i m} \sin 2 \Omega\right) h_{v} h_{w}^{2} / 4 \\
& +k \omega_{r}^{2}\left(\xi_{6 i m} \cos \eta+\xi_{6 r e} \sin \eta\right) \tag{18}
\end{align*}
$$

$$
h_{v} \dot{\eta}=h_{v} \sigma-\left(\xi_{4 i m}+\xi_{5 i m}\right) h_{v}^{3} / 4
$$

$$
-\left(\xi_{4 i m}+\xi_{5 i m} \cos 2 \Omega+\xi_{5 r e} \sin 2 \Omega\right) h_{v} h_{w}^{2} / 4
$$

$$
\begin{equation*}
+k \omega_{r}^{2}\left(\xi_{6 r e} \cos \eta-\xi_{6 i m} \sin \eta\right) \tag{19}
\end{equation*}
$$

$$
\begin{align*}
\dot{h_{w}} & =-\omega_{i} h_{w}+\left(\xi_{4 r e}+\xi_{5 r e}\right) h_{w}^{3} / 4  \tag{20}\\
& +\left(\xi_{4 r e}+\xi_{5 r e} \cos 2 \Omega+\xi_{5 i m} \sin 2 \Omega\right) h_{w} h_{v}^{2} / 4
\end{align*}
$$

$$
h_{v} \dot{\Omega}=\left\{-\xi_{5 r e} \sin 2 \Omega+\xi_{5 i m}(\cos 2 \Omega-1)\right\} h_{v}^{3} / 4
$$

$$
+\left\{-\xi_{5 r e} \sin 2 \Omega-\xi_{5 i m}(\cos 2 \Omega-1)\right\} h_{v} h_{w}^{2} / 4
$$

$$
\begin{equation*}
+k \omega_{r}^{2}\left(\xi_{6 r e} \cos \eta-\xi_{6 i m} \sin \eta\right) \tag{21}
\end{equation*}
$$

where $\xi_{j}=\xi_{j r e}+i \xi_{j i m}(j=4,5,6)$.
It is possible from Eqs. (18) through (21) to investigate many kinds of phenomena caused by the horizontal excitation
at the upper end of a cantilevered pipe. However, as a first step to clear such phenomena, this paper focuses on the effect of the forced excitation, whose frequency $v$ is near the second natural frequency $\omega_{r}$, and on the nonplanar flow-induced pipe vibration.

Equations (18) through (21) are also rewritten in the Cartesian forms in order to obtain the steady-state solutions $h_{v s}, h_{w s}, \eta_{s}$ and $\Omega_{s}$ and to investigate their stabilities. Defining $p_{1}=h_{v} \cos \eta$, $p_{2}=h_{v} \sin \eta, p_{3}=h_{w} \cos \Omega, p_{4}=h_{w} \sin \Omega$, equations with respect to $p_{1}, p_{2}, p_{3}$ and $p_{4}$ are obtained as follows:

$$
\begin{align*}
\dot{p_{1}}= & -\omega_{i} p_{1}-\sigma p_{2}+\frac{p_{1}^{2}+p_{2}^{2}}{4}\left(\xi_{3 r} p_{1}+\xi_{3 i} p_{2}\right) \\
& +\frac{p_{3}^{2}+p_{4}^{2}}{4}\left(\xi_{4 r} p_{1}+\xi_{4 i} p_{2}\right)+\frac{\xi_{5 r}}{4}\left(p_{3}^{2} p_{1}-p_{1} p_{4}^{2}+2 p_{3} p_{4} p_{2}\right) \\
& +\frac{\xi_{5 i}}{4}\left(p_{3}^{2} p_{2}-p_{4}^{2} p_{2}-2 p_{3} p_{4} p_{1}\right)+\xi_{6 i} k N^{2}  \tag{22}\\
\dot{p_{2}}= & -\omega_{i} p_{2}+\sigma p_{1}+\frac{p_{1}^{2}+p_{2}^{2}}{4}\left(\xi_{3 r} p_{2}-\xi_{3 i} p_{1}\right) \\
& +\frac{p_{3}^{2}+p_{4}^{2}}{4}\left(\xi_{4 r} p_{2}-\xi_{4 i} p_{1}\right)+\frac{\xi_{5 r}}{4}\left(p_{3}^{2} p_{2}-p_{4}^{2} p_{2}-2 p_{3} p_{4} p_{1}\right) \\
+ & +\frac{\xi_{5 i}}{4}\left(-p_{3}^{2} p_{1}+p_{4}^{2} p_{1}-2 p_{3} p_{4} p_{2}\right)+\xi_{6 r} k N^{2}  \tag{23}\\
\dot{p_{3}=}= & -\omega_{i} p_{3}+\frac{\xi_{3 r}}{4}\left(p_{3}^{2}+p_{4}^{2}\right) p_{3}+\frac{\xi_{4 r}}{4}\left(p_{1}^{2}+p_{2}^{2}\right) p_{3} \\
& +\frac{\left(p_{1}^{2}+p_{2}^{2}\right)}{4}\left(\xi_{5 r} p_{3}+2 \xi_{5 i} p_{4}\right)+\frac{\xi_{5 r}}{2} p_{3} p_{4}^{2}-\frac{\xi_{5 i}}{2} p_{4}^{3} \\
& -\frac{k N^{2}}{p_{1}^{2}+p_{2}^{2}}\left(\xi_{6 r} p_{1} p_{4}-\xi_{6 i} p_{2} p_{4}\right)  \tag{24}\\
\dot{p_{4}}= & -\omega_{i} p_{4}+\frac{\xi_{3 r}}{4}\left(p_{3}^{2}+p_{4}^{2}\right) p_{4}+\frac{\xi_{4 r}}{4}\left(p_{1}^{2}+p_{2}^{2}\right) p_{4} \\
& -\frac{\xi_{5 r}}{4}\left(p_{1}^{2}+p_{2}^{2}+2 p_{3}^{2}\right) p_{4}-\frac{\xi_{5 i}}{2} p_{4}^{2} p_{3} \\
& +\frac{k N^{2}}{p_{1}^{2}+p_{2}^{2}}\left(\xi_{6 r} p_{1} p_{3}-\xi_{6 i} p_{2} p_{3}\right) \tag{25}
\end{align*}
$$

Finally the equations of $\boldsymbol{y}_{i}(i=v, w)$ are obtained by the projection $Q_{i}=I-P_{i}$ where $I$ is an unit matrix. However $\boldsymbol{y}_{i}$, which are the elements of $M$ spanned by the linear stable vibration modes and become zero with time, are omitted in this paper. Therefore the pipe deflections $v$ and $w$ are expressed as follows:

$$
\begin{gather*}
v(s, t)=h_{v}\left|\Phi_{v 1}\right| \cos \left(\omega_{r} t+\phi+\angle \Phi_{v 1}\right)  \tag{26}\\
w(s, t)=h_{w}\left|\Phi_{w 1}\right| \cos \left(\omega_{r} t+\psi+\angle \Phi_{w 1}\right) \tag{27}
\end{gather*}
$$

by neglecting the terms which are equal to or higher than the third order with respect to $h_{v}$ and $h_{w}$. $\left|\Phi_{i 1}\right|$ and $\angle \Phi_{i 1}$ in equations (26) and (27), which are functions of $s$, are the magnitudes and arguments of the complex eigenfunctions $\Phi_{i 1}(i=v, w)$, respectively.

### 4.2 Case without Forced Excitation

In the case without forced excitation, i.e. $k=0$, there are two types of the steady-state solutions in Eqs. (18), (20) and (21) under the condition that the flow velocity $V_{s}$ is slightly over the critical value $V_{c r}$. Those solutions, which correspond to planar and nonplanar flow-induced vibrations of a cantilevered pipe with an end mass, are basically the same as those which were already clarified by Bajaj and Sethna (1984), and Troger and Steindl (1991) for the case of the cantilevered pipe with no end mass.

In the next section, we examine the effect of forced excitation on the stable steady-state nonplanar vibration of a cantilevered pipe with an end mass, where the amplitudes of $a_{v}$ and $a_{w}$ are

$$
\begin{equation*}
h_{v s}=h_{w s}=\sqrt{2 \omega_{i} / \xi_{4 r e}} \equiv h_{n p}, \tag{28}
\end{equation*}
$$

the phase difference $\Omega$ between $a_{w}$ and $a_{v}$ is

$$
\begin{equation*}
\Omega_{s}=\pi / 2 \tag{29}
\end{equation*}
$$

and the nonlinear frequencies of $v$ and $w$ are

$$
\begin{equation*}
\dot{\phi}_{s}=\dot{\psi_{s}}=\omega_{r}+\xi_{4 i m} h_{n p}^{2} / 2 \equiv \omega_{n p}, \tag{30}
\end{equation*}
$$

which are derived from Eqs. (18), (20) and (21) for the case of $k=0, \alpha=0.13, \beta=0.26$ and $\gamma=20.6$ (Yoshizawa et al,1998).

The transient time histories of $h_{v}, h_{w}$ and $\Omega$ of nonplanar flow-induced pipe vibration for $k=0$ and $V_{s}=6.50\left(v_{s}=6.40\right.$ $\mathrm{m} / \mathrm{s}$ ) are calculated numerically from equations (18), (20) and (21). As a result, the steady-state pipe motion in a horizontal plane at $s=0.7$, i.e. the dimensional distance from the upper end of the pipe 360 mm , takes on a circular shape, as shown in Fig. 3.

### 4.3 Case with Forced Excitation

In the case with forced excitation, i.e. $k \neq 0$, the steadystate solutions $p_{1 s}, p_{2 s}, p_{3 s}$ and $p_{4 s}$ are obtained by substituting $\dot{p}_{1}=\dot{p}_{2}=\dot{p}_{3}=\dot{p}_{4}=0$ into Eqs. (22) through (25). Then, we introduce $h_{v s}=\sqrt{p_{1 s}^{2}+p_{2 s}^{2}}, \eta=\tan ^{-1} p_{1 s} / p_{2 s}, h_{w s}=\sqrt{p_{3 s}^{2}+p_{4 s}^{2}}$ and $\Omega=\tan ^{-1} p_{3 s} / p_{4 s}$. Furthermore, $p_{1}, p_{2}, p_{3}$ and $p_{4}$ of Eqs. (22) through (25) are assumed as follows:

$$
\begin{gather*}
p_{1}=p_{1 s}+p_{1 d}(t), p_{2}=p_{2 s}+p_{2 d}(t) \\
p_{3}=p_{3 s}+p_{3 d}(t), p_{4}=p_{4 s}+p_{4 d}(t) \tag{31}
\end{gather*}
$$

in order to examine the stability of $p_{1 s}, p_{2 s}, p_{3 s}$ and $p_{4 s}$, i.e. $h_{v s}$, $h_{w s}, \eta_{s}$ and $\Omega_{s}$ for the case of nonplanar pipe vibration. Substituting equations (31) into equations (22) through (25), the linear
differential equations of $p_{1 d}, p_{2 d}, p_{3 d}$ and $p_{4 d}$ are derived by keeping only the linear terms with respect to $p_{1 d}, p_{2 d}, p_{3 d}$ and $p_{4 d}$. The stability of the steady-state solutions $p_{1 s}, p_{2 s}, p_{3 s}$ and $p_{4 s}$, i.e. $h_{v s}, h_{w s}, \eta_{s}$ and $\Omega_{s}$, is examined by solving numerically the characteristic equation, which is derived from the governing equations of $p_{1 d}, p_{2 d}, p_{3 d}$ and $p_{4 d}$. The stability of planar pipe vibration is also examined using equation (15) in a same manner.

Then, the frequency-response curves are obtained for primary resonance, i.e. $v=\omega_{r}+\sigma$ and $|\sigma| \ll \omega_{r}$, as shown in Fig.4. The solid and the broken lines correspond to the stable focuses and unstable saddle points, respectively, for $k=0.0058,0.0077$ and 0.0097 . The flow velocity $V_{s}$ is 6.50 .

Figure 4 shows that the steady-state planar pipe vibration is stable in the vicinity of $\sigma=0$. The frequency-response equation (Nayfeh and Mook, 1979) [12] for such a planar pipe vibration is derived from Eqs. (18) and (19) as follows:

$$
\begin{align*}
& \left\{\omega_{i}-\left(\xi_{4 r e}+\xi_{5 r e}\right) h_{v s}^{2} / 4\right\}^{2}+\left\{-\sigma+\left(\xi_{4 i m}+\xi_{5 i m}\right) h_{v s}^{2} / 4\right\}^{2} \\
& =k^{2} \omega_{r e}^{4}\left|\xi_{6}\right|^{2} / h_{v s}^{2} \tag{32}
\end{align*}
$$

It should be noted that the component $w$ of the nonplanar pipe vibration is diminished by the forced excitation which is perpendicular to $w$. This phenomenon is explained as follows. The third term in the right-hand side of equation (20), which is proportional to $h_{v}^{2}$, is caused by the third order nonlinear terms with two powers of $v$, for an example $w^{\prime} v^{\prime} v^{\prime \prime}$, in equation (2). We suppose that those terms act on the pipe motion in $z-x$ plane like a parametric excitation, and diminishes $w$. Then the steady-state amplitude of $h_{v}$ is determined by the first, second and 4th terms in the right-hand side of equation (18).

In Figure 4(a) and 4(b), in the case when $\sigma$ is relatively large, nontrivial steady-state amplitude in the $\omega$-direction exists and the stable steady-state nonplanar vibration occurs. It is noticed in this excitation frqeuncy region that the nonlinear characteristic of the response amplitude in the $v$-direction is changed from hardening-type to softening-type with decrease of the excitation frequency. Next let us decrease the excitation frequency. There is only the stable trivial steady-state amplitude in $w$-direction and the nonplanar vibration changes to the planar vibration, i.e., the nonplanar vibration is quenched to planar vibration. It is also seen in this region that the nonlinear characteristics of the response amplitude is hardening-type. More decrease of the excitation frequency makes both the stable nontrivial steady state in $v$-direction and the stable trivial steady state in $w$-direction destabilize. In this range, the natural frequency of the pipe and the excitation frequency do not match. This can be considered as a cause for the beat phenomenon. In this paper, we focus on the stable steady states of nonplanar and planar vibrations.

Next, we discuss the response frequency. Figure 4(c) shows the steady states of the parameter $\eta(=\sigma t-\phi)$ to determine the response frequency. In both cases of planar and non-planar vi-


FIGURE 3. The steady-state pipe motion in a horizontal plane without forced excitations ( $\alpha=0.13, \beta=0.26, \gamma=20.6, V_{s}=6.5$ )
brations, because $\eta(=\sigma t-\phi)$ has the stable steady state and is constant for each $\sigma$, the response frequency is quenched to the excitation frequency $v$ from Eq.(16).

Figure 4(d) shows the phase difference of the vibrations in the $v$ and $\omega$-directions $\Omega(=\psi-\phi)$ in the nonplanar vibration and these steady state of $\Omega$ is stable. While in the case without forced excitation, the phase difference $\Omega$ is equal to $\pi / 2$, the decrease of the excitation frequency increases $\Omega$ and makes the elliptic motion flat.

## 5 EXPERIMENT

### 5.1 Experimental Apparatus

The experimental setup is shown in Fig.5. The experiments were conducted with the silicone rubber pipe of 12 mm external diameter, 7 mm internal diameter and 518 mm length. The total mass $m l$ of the pipe is 44.6 g . The equivalent bending rigidity $E I$, which is estimated using the natural frequency 1.02 Hz of the first vibration mode at $v_{s}=0 \mathrm{~m} / \mathrm{s}$, is $0.01 \mathrm{~N} \mathrm{~m}^{2}$. The mass $M$ of the steal ring, which is fixed at the end of the pipe and corresponds to the end mass, is 44.6 g . The mass density $\rho$ of the water is $1.0 \mathrm{~g} / \mathrm{cm}^{3}$.

The values of $\alpha, \beta$ and $\gamma$ were determined experimentally as $0.13,0.26$ and 20.6, respectively.

The spatial displacements of the flexible pipe were measured by the image processing system which can perform measurements of the marker in three dimensional space, based on the images from two CCD cameras (OKK Inc., Quick MAG System).


FIGURE 4. Frequency-response curves obtained for primary resonance, i.e. $v=\omega_{r}+\sigma$ and $|\sigma| \ll \omega_{r}$ :(a) the amplitude $h_{v}$ of $a_{v}$,(b) the amplitude $h_{w}$ of $a_{w}$, (c) $\eta(=\sigma t-\phi)$, and (d) the phase difference $\Omega$ between $a_{v}$ and $a_{w} \cdot\left(\alpha=0.13, \beta=0.26, \gamma=20.6, V_{s}=6.5, k=0.0077\right)$

### 5.2 Experimental Results

The experimental critical flow velocity of flow-induced pipe vibration is $5.4 \mathrm{~m} / \mathrm{s}\left(V_{c r}=5.5\right)$ in the case of non-forced excitation. Although it was predicted theoretically that the nonplanar pipe vibration occurred over the critical velocity $V_{c r}$ at first, the stable planar vibration of the pipe was observed at the flow velocity $v_{s}=5.5 \mathrm{~m} / \mathrm{s}$ in case of non-forced excitation $\delta Y=0(k=0)$. We suppose that this discrepancy between the theoretical and experimental results is due to the different bending stiffnesses in two perpendicular directions (Bajaj and Sethna,1991).

The stable nonplanar vibration of the pipe was observed at the flow velocity $v_{s}=5.9 \mathrm{~m} / \mathrm{s}\left(V_{s}=6.6\right)$, as shown in Fig.6. Figure 6 shows the time histories of $v$ and $w$, their spectrum analyses and the pipe motions in a horizontal $y-z$ plane at $s=518 \mathrm{~mm}$ ( $s^{*}=0.7$ ).

The steady-state single-mode vibration of $v$ and $w$ is selfexcited with the natural frequency 2.13 Hz of the second vibration mode. As a result, the pipe motion in a horizontal plane at $s=518 \mathrm{~mm}$, takes on a circular shape broadly. The experimental result confirmed qualitatively the feature of the theoretical result, which is shown in Fig. 3.

When the excitation frequency is greater than the natural frequency as in Fig. 7, an elliptical motion of the pipe was observed. Furthermore, it can be seen from the frequency responses that the excitation frequency is the same as the frequency of the pipe vibration, which is known as a characteristic of the quenching


FIGURE 5. Experimantal setup
phenomenon. On the other hand, in the case when the excitation frequency is less than the natural frequency as shown in Fig. 8, the motion of the pipe was observed to be planar, and as in the case of Fig.6, the characteristic of the quenching phenomenon was seen from the frequency response curves. The above experimental results qualitatively verify the theoretically predicted characteristics on quenching of nonplanar vibrations of a pipe conveying fluid.

It should be noted that the transition from the nonplanar motion to the planar motion is different from the quenching phenomenon which is easily predicted in the case of the planar flow-induced vibration with forced excitation (Nayfeh and Mook, 1979 ; Yoshizawa et al., 1988). That is, the lateral deflection of the pipe $w$ in $z-x$ plane is also reduced by the forced excitation in $y-z$ plane.

Furthermore, there are many kinds of pipe motion, i.e. the beat phenomena and the weakly periodic nonplanar vibration of the pipe as functions of the amplitude $\delta Y$ and the frequency $N$ of the forced excitation.


FIGURE 6. Time histories of $v$ and $w$, their spectrum analyses and the pipe motion in the $y$-z plane ( $\delta Y=0 \mathrm{~mm}$ )

## 6 CONCLUSION

We have studied the effect of forced excitation on the nonplanar flow-induced vibration of a cantilevered pipe, which is hung vertically with an end mass, from the viewpoint of nonlinear dynamics. That is, the upper end of the pipe is excited periodically in a horizontal plane. The fluid velocity is slightly over the critical value, above which the nonplanar pipe vibration is self-excited due to an internal flow.

First, the first-order ordinary differential equations governing the amplitudes and phases of $v$ and $w$, where $v$ is the lateral deflection of the pipe in the direction of the forced excitation and $w$ is the lateral deflection of the pipe perpendicular to $v$, have been derived from the nonlinear coupled integro-partial differential equations for the $v$ and $w$. Those four amplitude and phase equations are coupled through the nonlinear terms with respect to the amplitudes of $v$ and $w$. By solving them numerically, it has been clarified that the nonplanar flow-induced pipe vibration is reduced to the planar vibration in the case of forced excitation, under the condition that the excitation frequency is nearly equal to the natural frequency of the flow-induced pipe vibration. Then, the lateral deflection $w$ of the pipe is reduced by the forced excitation perpendicular to $w$.

Second, the spatial behaviors of the silicon rubber pipe conveying water were observed quantitatively using the image processing system, which is based on the images from two CCD cameras. As predicted in the theory, the planar and nonplanar vibrations were observed depending on the forced excitation frequency.


FIGURE 7. Time histories of $v, w$ and $Y_{0}$, their spectrum analyses and the pipe motion in the $\mathrm{y}-\mathrm{z}$ plane $(\delta Y=4 \mathrm{~mm}, N=2.15 \mathrm{~Hz})$

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FIGURE 8. Time histories of $v, w$ and $Y_{0}$, their spectrum analyses and the pipe motion in the y -z plane $(\delta Y=4 \mathrm{~mm}, N=2.00 \mathrm{~Hz})$
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