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AN ANALYTICAL STUDY OF THE DYNAMICS OF PIPES CONVEYING FLUID OF AXIALLY VARYING DENSITY

Dana Giacobbi
McGill University
Montreal, Quebec, Canada
dana.giacobbi@mail.mcgill.ca

Christian Semler
McGill University, Simutech Group
Montreal, Quebec, Canada
csemler@simutechgroup.ca

Michael Païdoussis
McGill University
Montreal, Quebec, Canada
mary.fiorilli@mcgill.ca

ABSTRACT

This paper investigates the dynamics of a slender, flexible pipe, conveying a fluid whose density varies axially along the length of the pipe. Specific applications for this system have appeared in the mining of submerged methane crystals [1], but a general interest also exists due to more common situations in which fluid density changes along the length of the pipe, such as when a gas is conveyed at high velocity. Therefore, following a brief review of related work and of the well-established theory concerning pipes conveying fluid of constant density, the current problem is approached from an analytical perspective. In particular, a linear model describing the system is derived using a Hamiltonian approach, for the cases of (i) a pipe clamped at both ends and (ii) a cantilevered pipe, and results obtained using a Galerkin approach. Ultimately, it is shown that, in both the cantilevered and clamped-clamped cases, the behaviour of the system is similar to that of a pipe conveying fluid of constant density – that is, loss of stability by flutter and buckling respectively – save for two crucial differences. The first and most important is that it is the density at the discharging end which has the most significant effect on the critical flow velocities, rather than any other. Second, in the case of a cantilevered pipe, the magnitude of the density change can strongly influence in which mode the system loses stability, thereby also impacting the critical flow velocities. The specifics of both these effects are addressed in the paper.

1. INTRODUCTION

The dynamics of pipes conveying fluid has been studied extensively, with several hundreds of publications having been produced over the last 50 years, and still many more emerging each year [2]. Virtually inexhaustible interest in the problem exists for the reason that the pipe conveying fluid has become a paradigm in dynamics [3], the understanding of which radiates to many dynamical problems across Applied Mechanics. A review of some of the more recent developments and applications with regards to cantilevers in particular was done by Païdoussis [4]. From a broader perspective, some time ago, a review on the more general literature on axially moving continua was done by Wickert & Mote [5].

In this paper, the system at hand can be thought of in two ways, depending on one's preference: it is a pipe conveying fluid where either (i) the density of the fluid is changing at a specified rate as it travels through the pipe, or (ii) the fluid is accelerating axially inside the pipe, again at a controlled rate. In the circumstances being considered, the two are generally synonymous; however, depending on how one perceives the problem, the previous literature consulted might be different! In either case, the formulation investigated in the present paper has not been considered before, although several groups have carried out relevant work on related topics.

From the first perspective, there has been some – although relatively limited – research conducted on the dynamics of pipes or tubes conveying a *compressible* fluid, which has some implications in common with the system under consideration. Johnson *et al.* [6] appear to have been the first to consider the effects of compressibility on a cantilevered Euler-Bernoulli beam (or pipe) conveying fluid, concluding that compressibility can have a significant impact on its stability. Later, Johnson [7] studied the effect of fluid compressibility on critical velocity further, for a simply-supported Timoshenko pipe and a uniform isentropic flow model.

Approaching the problem from the second angle, a broader range of papers emerge, many falling outside the area of pipes conveying fluid. Some of the earliest work on axially accelerating continua was done by Miranker [8] who derived the equation of motion for an axially accelerating string, and

Mote [9] who obtained an approximate solution for such a string when it is harmonically driven at one end. More recently, a system much closer to the present one was considered by Lee [10] who looked at the effect of an accelerating mass on the vibration of a Timoshenko beam, and also by Wang [11], who obtained analytical and numerical stability for a similar system. Finally, the dynamic stability of pipes with harmonically oscillating fluid has been studied extensively, as summarized by Païdoussis [12].

The present work investigates the dynamics of pipes conveying fluid, where the fluid is incompressible, but of a specified axially varying density. Though an idealization, incompressibility is assumed, to narrow down the focus of the analysis to the exclusive effect of density change, without considering the added complexities of compressibility. Similarly, possible temporal variations in density have also been ignored, in order to decouple entirely the dynamical effect of density variation in space, and analyze it accordingly.

In this context, two sets of boundary conditions are considered for the pipe: in the first case both ends are clamped, but the discharging end is free to slide axially, and in the second case the pipe is cantilevered, i.e. one end is clamped and the other is free. One specific application of the cantilevered case exists in the exploitation of the all-important submerged methane crystals [1], for which one design would call for a long flexible pipe to aspirate the methane. In such a design, the methane would likely undergo one or more phase changes, bringing about a varying axial fluid density. In addition and perhaps more importantly, one can think of any number of mechanisms for generating variable fluid density along the pipe: heating, pressure loss, or even compressibility, in which understanding the effect of axial fluid density change on stability would be important.

Following a brief review of the general theory of pipes conveying fluid, this paper presents a new theory for pipes conveying incompressible fluids with axially varying density. A Hamiltonian derivation is employed, and a linear equation of motion obtained and rendered non-dimensional. Finally, solutions are obtained using a Galerkin method, as proposed by Païdoussis [12], and results are presented in the form of critical flow velocities, critical frequencies and Argand diagrams.

2. BASIC THEORY

The simplest equation governing the motion of a pipe conveying fluid, represented in Fig. 1, is given by

$$EI\frac{\partial^4 w}{\partial x^4}+MU^2\frac{\partial^2 w}{\partial x^2}+2MU\frac{\partial^2 w}{\partial x\partial t}+(M+m)\frac{\partial^2 w}{\partial t^2}=0, \qquad (1)$$

where EI is the flexural rigidity, w is the lateral displacement, x is the axial coordinate, M is the fluid mass per unit length, U is the fluid flow velocity, t is time, and m is the pipe mass per unit length. In order of appearance, the terms of the equation are the flexural restoring force, the flow-related centrifugal force, the flow-related Coriolis force, and the inertial force. It is worth noting that frictional forces are not neglected in this equation, but rather that viscous traction on the pipe and viscous pressure-loss forces cancel out in the linear limit in the process of deriving this equation [12].

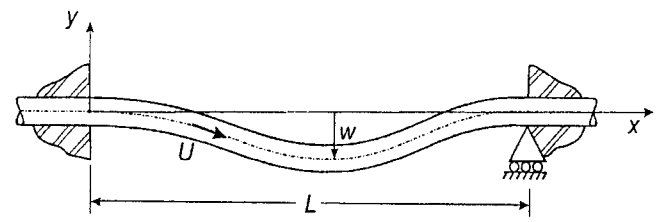


Figure 1. Diagram of a pipe conveying fluid, positively clamped at the upstream end and clamped with axial sliding permitted at the downstream end.

For the simplest case of a pipe of length, L, discharging fluid as described above, the work done by the fluid on the pipe over a cycle of oscillation is

$$\Delta W=-MU\int_0^T\left[\left(\frac{\partial w}{\partial t}\right)^2+U\left(\frac{\partial w}{\partial t}\right)\left(\frac{\partial w}{\partial x}\right)\right]_0^L \mathrm{d}t. \qquad (2)$$

Clearly, in the case of supported ends, the work done by the fluid must be zero (since $\partial w/\partial t$ at both ends is zero), and no dynamic instability can arise. However, careful consideration of Eq. (1) shows that, even in the absence of motion, the centrifugal force can be treated as a compressive force, resulting in the system losing stability by static divergence, or buckling. In the case of a cantilevered pipe, however, Eq. (2) reduces to

$$\Delta W=-MU\int_0^T\left[\left(\frac{\partial w}{\partial t}\right)_L^2+U\left(\frac{\partial w}{\partial t}\right)_L\left(\frac{\partial w}{\partial x}\right)_L\right]\mathrm{d}t\neq 0. \qquad (3)$$

In this case, it is plausible that, for large enough U, the amount of work done by the fluid on the pipe could become positive, thus feeding energy into the system and creating the potential for a dynamic instability. This, indeed, is exactly what happens, as flutter has long since been demonstrated by experiments and theory [12-14].

3. DERIVATION FOR A PIPE CONVEYING FLUID OF AXIALLY VARYING DENSITY

3.1 HAMILTONIAN DERIVATION OF THE EQUATION OF MOTION

The derivation for the pipe conveying incompressible fluid of specified axially varying density is done using a version of Hamilton's principle modified for open systems, and is very similar to the derivation by Païdoussis [12] for a pipe conveying a fluid of constant density. The derivation is carried out with the cantilevered and clamped-clamped boundary conditions in mind, but any set of boundary conditions for which the pipe displacement is zero at $x = 0$ is acceptable.

For an open system, Hamilton's principle can be expressed as

$$\delta \int_{t_1}^{t_2} \mathcal{L}_0 \mathrm{d}t + \int_{t_1}^{t_2} \delta H \mathrm{d}t = 0, \tag{4}$$

where

$$\delta H = \delta W + \iint_{S_0(t)} \rho (\mathbf{u} \cdot \delta \mathbf{r})(\mathbf{V} - \mathbf{u}) \cdot \mathbf{n} \, \mathrm{d}S \tag{5}$$

and in which $\mathcal{L}_0 = T_0 - V_0$ is the Lagrangian of the open system, δW is the virtual work done by the generalized forces, and ρ is the fluid density of fluid particles , each with position $\mathbf{r}$ and velocity $\mathbf{u}$. Furthermore, it is assumed that S_0 is a portion of the control surface of the open system, capable of movement with a velocity $\mathbf{V} \cdot \mathbf{n}$ normal to the surface, across which mass may be transported; $\mathbf{n}$ is the outward normal. For additional details and figures pertaining to Hamilton's principle as it applies to open systems, the reader is referred to Païdoussis [12].

This "modified" principle is next applied to the case of a pipe conveying an incompressible fluid. No boundary conditions need yet be imposed, but damping, gravity, external tension, external pressurization, and the effect of any external dense fluid have all been neglected for simplicity. However, the density is still assumed to be variable, and consequently so is the flow velocity. Finally, it is presumed that the only forces involved in δW are associated with the pressure, p, measured above the ambient of the surrounding medium; hence

$$\delta H = -\iint_{S_c(t)+S_i+S_e(t)} p(\delta \mathbf{r} \cdot \mathbf{n}) \mathrm{d}S + \iint_{S_i+S_e(t)} \rho (\mathbf{u} \cdot \delta \mathbf{r})(\mathbf{V} - \mathbf{u}) \cdot \mathbf{n} \, \mathrm{d}S, \tag{6}$$

where $S_c(t)$ is the surface covered by the pipe wall, and S_i and $S_e(t)$ are the respective inlet and exit open surfaces for the fluid. Next, it is presumed that virtual displacements of the pipe are independent of those of the fluid, and that the density of the fluid is an explicit function of x. Therefore, because the fluid is incompressible, there can be no virtual change in the volume of the system, and Eq. (6) can be rewritten as

$$\delta H = -\iint_{S_i+S_e(t)} p(\delta \mathbf{r} \cdot \mathbf{n}) \mathrm{d}S + \iint_{S_i+S_e(t)} \rho (\mathbf{u} \cdot \delta \mathbf{r})(\mathbf{V} - \mathbf{u}) \cdot \mathbf{n} \, \mathrm{d}S, \tag{7}$$

where $S_c(t)$ has been dropped from the surface of integration. Subsequently, it is assumed that the fluid entrance conditions remain prescribed and constant. Therefore, the integrals over S_i are zero, and so too is the first integral over $S_e(t)$, since at the outlet $p = 0$ is taken with no loss of generality. Therefore, for a pipe of length L, Eq. (7) reduces to

$$\delta H = \iint_{S_i+S_e(t)} \rho (\mathbf{u} \cdot \delta \mathbf{r})(\mathbf{V} - \mathbf{u}) \cdot \mathbf{n} \, \mathrm{d}S = -M_L U_L (\dot{\mathbf{r}} + U_L \boldsymbol{\tau}_L) \cdot \delta \mathbf{r}_\mathrm{L}, \tag{8}$$

in which the relations $\mathbf{u} = \dot{\mathbf{r}} + U\boldsymbol{\tau}$, $(\mathbf{u} - \mathbf{V}) \cdot \mathbf{n} = U_L$ at $\ell_e(t)$ and $M = \rho A$ have been used, in which M is the fluid mass per unit length and is a function of x, and A is the fluid cross-sectional area and is a constant. However, even though the flow velocity and lineal density are each individually functions of x, their product, the mass flow rate MU, must remain constant, such that $M_L U_L = MU$. Therefore, by substituting Eq. (8) into Eq. (4), Hamilton's principle results in

$$\delta \int_{t_1}^{t_2} \mathcal{L}_0 dt - \int_{t_1}^{t_2} MU(\dot{\mathbf{r}} + U_L \boldsymbol{\tau}_L) \cdot \delta \mathbf{r}_\mathrm{L} \mathrm{d}t = 0. \tag{9}$$

Next, the equation of motion is derived using Eq. (9) and several useful relations developed by Païdoussis [12]. The pipe is assumed to be inextensible, and use is made of the curvilinear coordinate s, with this assumption, the following apply: (i) the axial displacement, u, is equal to $u = x - x_0$, with $x_0 = s$, and so $\dot{u} = \dot{x}$; (ii) $\partial x / \partial s = \left[1 - (\partial z / \partial s)^2\right]^{1/2}$, with $z = w$, and hence $\partial x / \partial s \approx 1 - \frac{1}{2} w'^2$, where $(\ )' = \partial(\ ) / \partial s$; (iii) $u_L = -\int_0^L \frac{1}{2} w'^2 \mathrm{d}s$. Furthermore, one may write $\dot{\mathbf{r}} = \dot{x}_L \mathbf{i} + \dot{z}_L \mathbf{k} = \dot{u}_L \mathbf{i} + \dot{w}_L \mathbf{k}$; $\boldsymbol{\tau} = x'_L \mathbf{i} + z'_L \mathbf{k} \approx \left[1 - \frac{1}{2} w'^2_L\right] \mathbf{i} + w'_L \mathbf{k}$; and finally $\delta \mathbf{r}_L = \delta u_L \mathbf{i} + \delta w_L \mathbf{k}$. It is important to mention that the inextensibility condition can only be satisfied if the pipe is free to move axially at $x = L$, and so this condition is likewise

assumed.[1] Therefore, after some manipulation, Eq. (9) may be rewritten as

$$\delta\int_{t_1}^{t_2}\left(\mathcal{L}_0 - MUU_L u_L\right)\mathrm{d}t - \int_{t_1}^{t_2} MU\left(\dot{w}_L + U_L w_L'\right)\cdot\delta w_L\,\mathrm{d}t = 0, \tag{10}$$

correct to $O(\varepsilon^2)$, and for which the Lagrangian must now be evaluated. Hence, the kinetic energies of the pipe and fluid, respectively, are expressed as

$$T_P = \tfrac{1}{2} m\int_0^L \dot{w}^2\mathrm{d}s, \tag{11}$$

$$T_f = \tfrac{1}{2}\int_0^L M\left[U^2 + \dot{w}^2 + 2U\dot{w}w' + 2U\dot{u}\right]\mathrm{d}s; \tag{12}$$

the potential energy of system, which in this case is equal to that of the pipe, is

$$V = V_P = \tfrac{1}{2} EI\int_0^L w''^2\mathrm{d}s. \tag{13}$$

Next, Eqs. (11) – (13) are substituted into Eq. (10), and standard variational techniques used to obtain the final form of each term. In particular, the term including the kinetic energy of the pipe becomes

$$\tfrac{1}{2}\delta\int_{t_1}^{t_2} m\int_0^L \dot{w}^2\mathrm{d}s\mathrm{d}t = -\int_{t_1}^{t_2}\int_0^L \ddot{w}\delta w\,\mathrm{d}s\mathrm{d}t, \tag{14}$$

and the term containing the potential energy of the pipe is

$$\delta\int_{t_1}^{t_2}\tfrac{1}{2} EI\int_0^L w''^2\mathrm{d}s\mathrm{d}t = -\int_{t_1}^{t_2}\int_0^L EIw''''\delta w\,\mathrm{d}s\mathrm{d}t. \tag{15}$$

Lastly, recalling that MU is a constant , and also that M and U are constant in time – though not in space – the more complicated term containing the kinetic energy of the fluid can be transformed, after considerable manipulation, as follows:

$$\begin{aligned}&\tfrac{1}{2}\delta\int_{t_1}^{t_2}\int_0^L M\left[U^2 + \dot{w}^2 + 2U\dot{w}w' + 2U\dot{u}\right]\mathrm{d}s\mathrm{d}t\\ &= -\int_{t_1}^{t_2}\int_0^L M\ddot{w} + 2MU\dot{w}'\delta w\,\mathrm{d}s\mathrm{d}t + \int_{t_1}^{t_2} MU\dot{w}\delta w\Big|_0^L.\end{aligned} \tag{16}$$

Next, recalling that $u_L = -\int_0^L \tfrac{1}{2}w'^2\mathrm{d}s$, the last remaining term containing δ in Eq. (10) may also be operated on, as follows:

$$\begin{aligned}&-\delta\int_{t_1}^{t_2} MUU_L u_L\,\mathrm{d}t\\ &= MUU_L\int_{t_1}^{t_2} w'\delta w\Big|_0^L\mathrm{d}t - MUU_L\int_{t_1}^{t_2}\int_0^L w''\delta w\,\mathrm{d}s\mathrm{d}t.\end{aligned} \tag{17}$$

Finally, by adding the contributions of Eqs. (14) – (17) into Eq. (10), the equation of motion can be derived. It can be seen that terms involving $(\delta w)_L$ cancel out, such that the only remaining single-integral terms involve $(\delta w)_0$. The boundary condition at $x = 0$ may now be invoked, and in both cases of interest, i.e. the cantilevered and clamped-clamped configurations, $(\delta w)_0 = 0$. Therefore, the equation of motion is given by

$$EIw'''' + 2MU\dot{w}' + MUU_L w'' + \left[M(x) + m\right]\ddot{w} = 0, \tag{18}$$

in which the fact that $M \equiv M(x)$ is stressed.

Perhaps not so surprisingly after the fact, the equation of motion is virtually the same as for a pipe conveying a fluid with constant density at constant flow velocity. Certainly, looking back at the starting point for the analysis (Eqs. (10-13)), and recalling that MU is a constant, it is clear that only the centrifugal and inertia terms were susceptible to change *vis-à-vis* the constant-density case. Nevertheless, the key distinctive characteristics of this equation are that the centrifugal term coefficient, MUU_L, must be evaluated at L, and the inertia term coefficient, $M(x) + m$, is a function of x.

3.2 NON-DIMENSIONALIZATION AND METHOD OF SOLUTION

Eq. (18) can be rendered non-dimensional by making use of the parameters

$$\xi = \frac{x}{L},\quad \eta = \frac{w}{L},\quad \tau = \left(\frac{EI}{M(0) + m}\right)^{1/2}\frac{t}{L^2}. \tag{19}$$

Moreover, rewriting $M(x) = M(\xi)$ and $M(L) = M(1)$, and defining the ratio of lineal fluid mass density along the pipe with respect to that at the inlet, $\mu(\xi) \equiv M(\xi)/M(0)$ – also noting that, at the outlet, $\mu(1) \equiv M(1)/M(0)$ – the non-dimensional equation can be expressed as

$$\frac{\partial^4\eta}{\partial\xi^4} + 2\beta^{1/2}u\frac{\partial^2\eta}{\partial\xi\partial\tau} + \mu(1)^{-1}u^2\frac{\partial^2\eta}{\partial\xi^2} + \left\{1 - \left[\beta(1 - \mu(\xi))\right]\right\}\frac{\partial^2\eta}{\partial\tau^2} = 0, \tag{20}$$

in which two non-dimensional parameters have been utilized:

[1] The case an extensible pipe with no motion at $x = L$ will follow a distinct but similar derivation that is not presented here. As discussed by Païdoussis [12], the final equation of motion will, in fact, be identical; however, certain terms arise in a different manner.

$$u=\left(\frac{M(0)}{EI}\right)^{1/2}U(0)L,\quad \beta=\frac{M(0)}{M(0)+m}\ , \tag{21}$$

which are similar to those for the case of constant density. In the foregoing, it must be noted that this selection of non-dimensional parameters is one of many potential choices; an alternative possibility involves making use of $M(L)$ and $U(L)$ instead of $M(0)$ and $U(0)$. Nevertheless, the current set is arguably the most intuitive, and definitely the easiest choice for comparing results with the case of a pipe conveying fluid of constant density. Finally, though it does not appear in the equation of motion explicitly, the dimensional frequency Ω, in radians per second, is related to the dimensionless one as follows:

$$\omega=\left(\frac{M(0)+m}{EI}\right)^{1/2}\Omega L^2. \tag{22}$$

In general, $\mu(\xi)$ is arbitrary, and Eq. (20) does not have constant coefficients. Therefore, the Galerkin procedure proposed by Païdoussis [12] is advised in order to obtain approximate solutions. We assume a solution of the form

$$\eta(\xi,\tau)=\sum_{r=1}^{N}\phi_r(\xi)q_r(\tau), \tag{23}$$

where $\phi_r(\xi)$ are the comparison functions, for which the eigenfunctions of the pipe (beam) are an obvious choice, and $q_r(\tau)$ are the generalized coordinates. Substituting Eq. (23) into Eq. (20) results in

$$\sum_{r=1}^{N}\left\{\begin{array}{l}\lambda_r^4\phi_r q_r+\mu(1)^{-1}u^2\phi_r''q_r\\ +2\beta^{1/2}u\phi_r'\dot{q}_r+\{1-[\beta(1-\mu(\xi))]\}\phi_r\ddot{q}_r\end{array}\right\}=0, \tag{24}$$

where for economy of writing, the prime and overdot stand for $\partial(\ )/\partial\xi$ and $\partial(\ )/\partial\tau$. Furthermore, by pre-multiplying Eq. (24) by $\phi_s(\xi)$ and integrating over the domain, i.e. $[0,1]$, the equation can be decoupled, as follows:

$$\lambda_r^4\delta_{sr}q_r+\mu(1)^{-1}u^2c_{sr}q_r+2\beta^{1/2}ub_{sr}\dot{q}_r+\mu_{sr}\ddot{q}_r=0, \tag{25}$$

where $\delta_{sr}=\int_0^1\phi_s\phi_r\,\mathrm{d}\xi$ and is the Kronecher delta, $b_{sr}=\int_0^1\phi_s\phi_r'\,\mathrm{d}\xi$, $c_{sr}=\int_0^1\phi_s\phi_r''\,\mathrm{d}\xi$, and

$$\mu_{sr}=\int_0^1\{1-[\beta(1-\mu(\xi))]\}\phi_s\phi_r\,\mathrm{d}\xi. \tag{26}$$

In the above, if $\phi_r(\xi)$ are the eigenfunctions of the pipe, Païdoussis [12] provides exact values for the constants b_{sr} and c_{sr}; μ_{sr} in (26) should not be confused with $\mu(\xi)$ in (20). Moreover, if $\mu(\xi)$ is also known, Eq. (25) can be solved by standard ordinary differential equations methods. In particular, two simple but representative density distributions are the case of (i) a step-distribution, where the density changes abruptly halfway through the pipe, and (ii) a ramp distribution. In these two cases, Eq. (26) can be simplified even further, such that, for the step distribution

$$\left(\mu_{\text{step}}\right)_{sr}=\delta_{sr}+\beta[\mu(1)-1]\int_{1/2}^{1}\phi_s\phi_r\,\mathrm{d}\xi, \tag{27}$$

and for the ramp distribution

$$\left(\mu_{\text{ramp}}\right)_{sr}=\delta_{sr}+\beta[\mu(1)-1]\int_0^1\xi\phi_s\phi_r\,\mathrm{d}\xi. \tag{28}$$

4. THEORETICAL RESULTS

Theoretical results have been obtained for a pipe conveying an incompressible fluid of axially varying density. The two cases considered are a pipe that is clamped at both ends and one that is cantilevered. Of interest are the critical flow velocities for any instability that may arise, plotted versus the mass ratio $\mu(1)$, defined above Eq. (20). In addition, in the case of the cantilevered pipe, the general frequency characteristics are discussed with reference to representative Argand diagrams, and the frequencies of oscillation at the threshold of flutter are plotted versus $\mu(1)$.

4.1 RESULTS FOR THE CLAMPED-CLAMPED PIPE

For a pipe clamped at both ends (though with one end free to slide axially), the arguments summarized in Section 2 indicate that a static instability is the only one possible. Therefore, it is feasible to obtain a solution to Eq. (25) by considering exclusively the static terms, i.e. the flexural restoring force and the centrifugal force. In doing so, it becomes obvious that the result will be exactly the same as for the pipe conveying fluid of constant density, except for a multiplicative factor related to $\mu(1)$, as follows:

$$u_{\text{crit}}=2\pi\sqrt{\mu(1)}. \tag{29}$$

In particular, the parameters β and μ_{sr} have no influence on the stability of the system.

With this done, it is always of interest to verify that the full solution recovers the simpler one, and so the stability of the clamped-clamped pipe was also studied using the full dynamic equation. In this case, four modes were used in the Galerkin analysis, and results obtained for several values of $\mu(1)$. A linear density distribution and several values of β were considered. As shown in Fig. 2, which plots u_{crit} against $\mu(1)$ for $\beta = 0.5$, the dynamic solution fully agrees with Eq. (29), regardless of β, and therefore also of μ_{sr}.

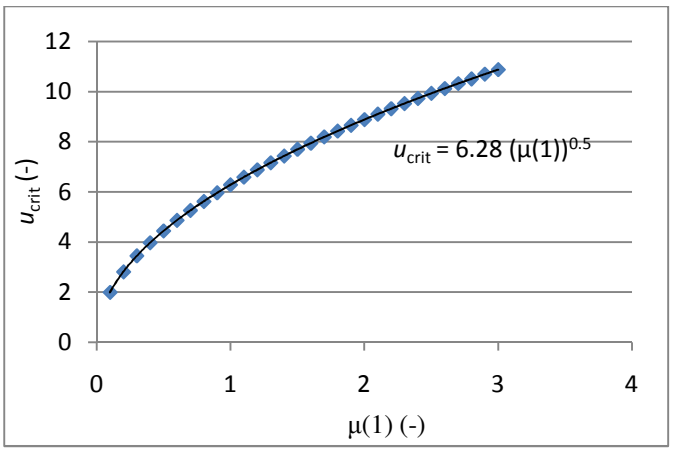


Figure 2. Dimensionless critical flow velocity plotted versus $\mu(1)$ for a pipe clamped at both ends.

4.2 RESULTS FOR THE CANTILEVERED PIPE

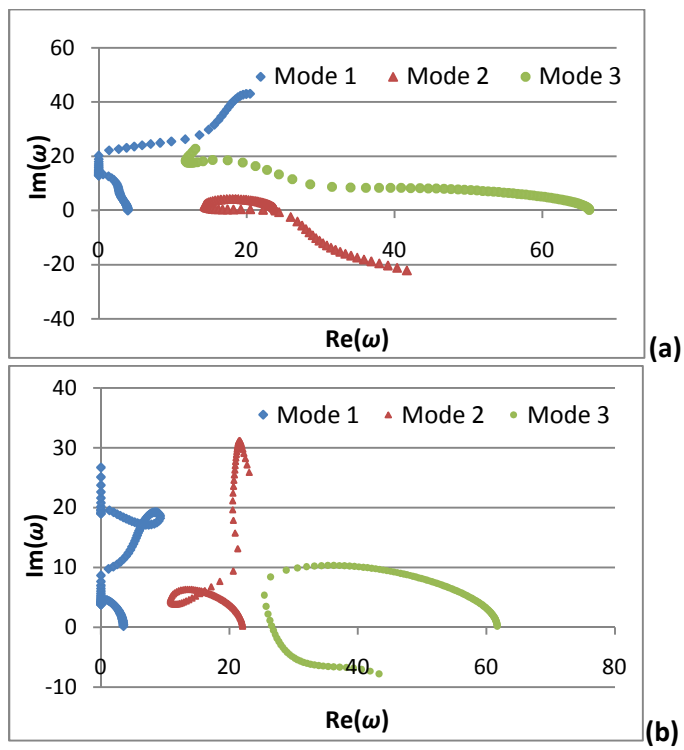
In the case of the cantilevered pipe, the solution is more complex, and therefore more interesting, as clearly the potential for flutter exists. However, by carefully examining the definition of μ_{sr} in Eq. (26), it becomes apparent that with decreasing β, the value of μ_{sr} approaches unity (or zero, if $s \neq r$) and the problem begins to resemble a constant density problem, but for the factor $\mu(1)$. Therefore, for small enough β, the critical flow velocity will be simply related to the constant density case, as follows:

$$u_{\text{crit}} = [u_{\text{crit}}]_{\mu(1)=1} \sqrt{\mu(1)}. \tag{30}$$

where $[u_{\text{crit}}]_{\mu(1)=1}$ is the non-dimensional critical flow velocity for a pipe conveying fluid of constant density.

More interesting, is a situation with large β. In this case, the expression for μ_{sr} can be complicated, and affects both the critical flow velocities and frequency characteristics of the system. Under these circumstances, it must be ensured that the number of modes used in the Galerkin analysis is sufficiently large, particularly for very large values of β and $\mu(1)$. The results presented here were obtained with a 6-mode Galerkin analysis; for larger values of $\mu(1)$, 8 modes or more become necessary.

In particular, Fig. 3 illustrates how, as $\mu(1)$ is increased, the frequency characteristics change drastically, and flutter can arise in different modes. In particular, for $\beta = 0.5$ and a linear density function, flutter arises in (a) the second mode for $\mu(1) = 0.5$, (b) in the third for $\mu(1) = 1.0$, and (c) in the first for $\mu(1) = 1.5$. It should be noted that case (b), $\mu(1) = 1.0$, is actually the constant density configuration, such that Fig. 3 is a good illustration of the important qualitative effect that density change has on the dynamics. Moreover, a change in the unstable mode is often, though not always, associated with a jump in the critical frequency, as illustrated by Fig. 4. However, the mode change and frequency jump do not occur for precisely the same $\mu(1)$. Additionally, for large β, Eq. (30) no longer holds true. The critical flow velocity is affected in various ways, and follows the same "jumps" as the critical frequency, but less pronounced, as illustrated by Fig. 5. Similarly to Fig.3, in both Figs. 4 and 5, the constant density case, $\mu(1) = 1.0$, lies at the centre of the graph as a point of reference. Altogether, it is clear that the change in density, as well as the magnitude and nature of that change, play a significant role in the dynamics of the system.



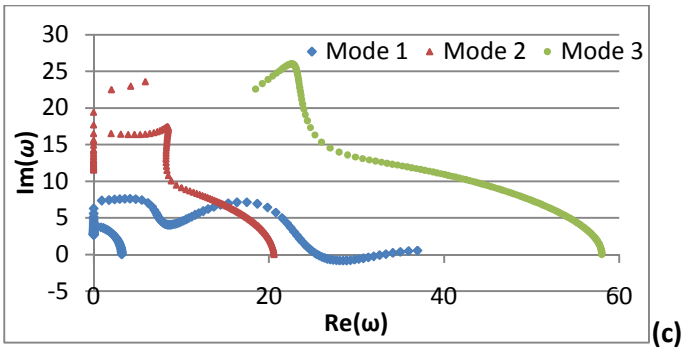


Figure 3. Argand diagrams for a cantilevered pipe conveying fluid of axially varying density, for $\beta = 0.5$ and a ramped density function: **(a)** $\mu(1) = 0.5$; **(b)** $\mu(1) = 1.0$; **(c)** $\mu(1) = 1.5$.

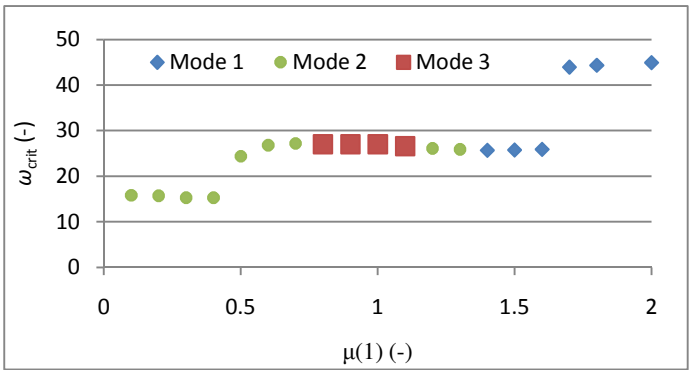


Figure 4. Dimensionless frequencies plotted against $\mu(1)$ for a cantilevered pipe conveying fluid of axially varying density, for $\beta = 0.5$ and a linear density function.

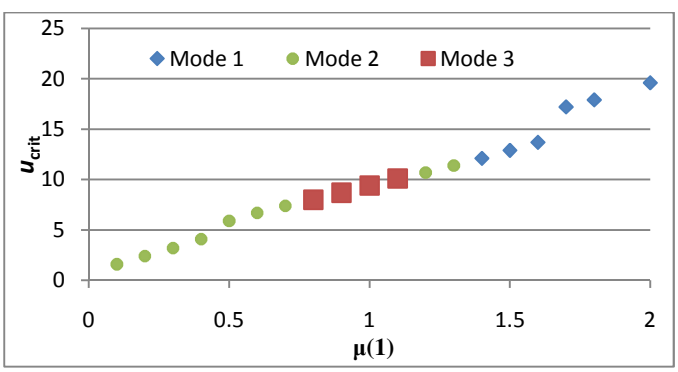


Figure 5. Dimensionless critical flow velocities plotted against $\mu(1)$ for a cantilevered pipe conveying fluid of axially varying density, for $\beta = 0.5$ and a linear density function.

5. CONCLUSION

This paper presented an analytical investigation of the dynamics of a pipe conveying an incompressible fluid, where the density of the fluid varies axially along the length of the pipe. A new linear equation of motion has been derived, and the dynamical behaviour investigated for a pipe clamped at both ends and for a cantilevered pipe.

The effect of varying density was found to be significant: in both cases it is the flow velocity at the discharging end which affects the dynamics through the centrifugal term; it is therefore important to account for the density change as prescribed in the new equation of motion, rather than taking, for example, an average density. Moreover, in the case of the cantilevered pipe, it was found that the change in density further affects the dynamics through the inertial term, for a heavy enough fluid.

Finally, though the analytical results obtained can be utilized with confidence, corroboration would undoubtedly be desirable. However, in experiments, it could be quite difficult to isolate the effect of density change, and even more difficult to specify that change explicitly. Therefore, a fully numerical approach has been initiated, wherein the numerical simulations essentially simulate experiments but with complete control over the density. The results of this corroboration are not yet complete, and will not be commented upon further here.

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