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PHASE-SHIFT DETERMINATION IN CORIOLIS FLOWMETERS WITH ADDED MASSES

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ABSTRACT

In this study, an approximate analytic solution for phase-shift (and thus mass flow) prediction along the length of the measuring tube of a Coriolis flowmeter is investigated. A single, straight measuring tube is considered; added masses at the sensor locations, are included in the model, and thus in the equation of motion. The method of multiple timescales, an approximate analytical technique, has been applied directly to the equation of motion, and the equations of order one and epsilon have been obtained analytically for the system at resonance. The solution of the equation of motion is obtained by satisfying the solvability condition (making the solution of order epsilon free of secular terms). The measuring tube is excited by the driver, and the phase-shift is measured at two symmetrically located points on either side of the mid-length of the tube. The effects of system parameters on the measured phase-shift are discussed.

1. INTRODUCTION

One prominent use of fluid-conveying pipes, for sub-critical flow velocities, is in Coriolis flowmetering applications. Over the last decades, Coriolis flowmeters have been used widely in industrial applications, e.g. in the food and beverage industries and oil, gas and chemical plants. This is because they are capable of measuring directly the true mass flow, with high accuracy, wide rangeability and high repeatability.

A Coriolis flowmeter consists of one or a number of measuring, fluid-conveying tubes, two sensors, a driver and housing. As the driver makes the measuring tube oscillate at a resonant frequency (usually in the lowest mode), the cross-sections of the tube rotate. Due to the resultant angular velocity, Coriolis forces arise, preventing standing (stationary) flexural waves, and thus a travelling-wave arises; i. e. there is a shift in the phase of the response of any two points along the tube. For a steady flow, this phase-shift is usually linearly proportional to the mass flow rate, and this can be used to determine the flow rate.

A detailed numerical model of a real Coriolis flowmeter has been developed in [1, 2], employing a finite element method, which could simulate all possible, important and practical features, and consequently predict the phase-shift. However, it is difficult to draw any general conclusions from these studies regarding the effects of practical parameters and conditions. Therefore, it is desirable to simplify the model so as to draw general conclusions and get direct insight regarding the basic relationships between the phase-shift and the parameters of the system. An analytical method may be a good candidate for this goal, and it can readily be applied to simplified mathematical models of Coriolis flowmeters.

The published information of the last decade on Coriolis flowmeters was reviewed by Anklin et al. [3], discussing factors and imperfections influencing phase-shift measurements. Some factors which can affect the phase-shift are: (i) any kind of "imperfection", e.g. damping [4, 5], external vibrations [6] due to the pumping system, neighbouring vibrating devices, and flow compressibility [7]; (ii) geometric or material nonlinearities; (iii) pulsating flow (the question being how pulsations cause errors in mean flow measurements [8, 9]).

In this paper, an approximate analytical solution for phase-shift prediction along the length of the measuring tube of a Coriolis flowmeter (and thus mass flow prediction) is investigated. A single, straight measuring tube is considered; added masses at the sensor and driver locations are included in the model, and thus in the equation of motion. The method of multiple timescales, an approximate analytical technique, will be applied directly to the equation of motion, and the equations of order 1 and ε will be obtained analytically for the system at primary resonance. Satisfying the solvability condition (making the solution of order ε free of secular terms), the vibration response is obtained.

Once the response of the measuring tube to the external excitation (exerted by the driver) is obtained via the above-mentioned technique, the phase-shift is determined at two symmetrically located points around the mid-tube. The effects of system parameters on the measured phase-shift are discussed.

2. MATHEMATICAL MODEL AND EQUATION OF MOTION

The tube assembly of a single-tube Coriolis flowmeter consists of a straight single fluid-conveying *measuring tube*, two sensors and a driver. This system is modelled here as a clamped-clamped straight fluid-conveying pipe with three point masses and an external force (Fig.1). The system consists of a pipe of length L, inner radius r, thickness h, cross-sectional area A, flexural rigidity EI, and mass per unit length m, conveying a fluid of mass M per unit length with velocity U. There is also a mass, m_s, attached to the pipe at distance x_{s1} from the left-hand end, and another at x_{s2}; these masses are simplified model of the sensors. The driver is modelled as an external force of amplitude F and frequency Ω, along with a point mass, m_f, located at a distance x_f from the left-hand end.

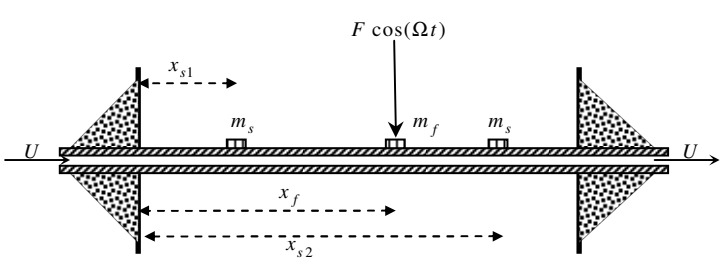

Figure 1: Schematic of a straight single-tube fluid-conveying measuring tube.

The equation of motion derived either by Newtonian mechanics or extended Hamilton's principle is [10]

$$EI\frac{\partial^4 w}{\partial x^4}+MU^2\frac{\partial^2 w}{\partial x^2}+2MU\frac{\partial^2 w}{\partial x\partial t}$$
$$+(M+m)\frac{\partial^2 w}{\partial t^2}+m_s\left[\delta\left(x-x_{s1}\right)+\delta\left(x-x_{s2}\right)\right]\frac{\partial^2 w}{\partial t^2}$$
$$+m_f\delta\left(x-x_f\right)\frac{\partial^2 w}{\partial t^2}=F\delta\left(x-x_f\right)\cos\left(\Omega t\right), \tag{1a}$$

where $w=w(x,t)$ is the transverse displacement of each point of the measuring tube from static equilibrium $w=0$, x the axial coordinate, t time, and $\delta(x)$ the Dirac delta function. In order to derive the above equation, the following assumptions have been made: (i) the pipe is straight, undamped, and perfectly clamped at both ends; (ii) the pipe is modelled as an Euler-Bernoulli beam, i.e., rotary inertia and shear deformation are neglected; (iii) the fluid is incompressible and the flow velocity is uniform with a constant velocity profile (plug-flow).

The boundary conditions for a clamped-clamped system are

$$w\big|_{x=0}=0,\ \ w\big|_{x=L}=0,\ \ \left.\frac{\partial w}{\partial x}\right|_{x=0}=0,\ \ \left.\frac{\partial w}{\partial x}\right|_{x=L}=0. \tag{1b}$$

With the aid of the dimensionless quantities

$$\xi=\frac{x}{L},\ \xi_{s1}=\frac{x_{s1}}{L},\ \xi_{s2}=\frac{x_{s2}}{L},\ \xi_f=\frac{x_f}{L},\ \eta=\frac{w}{L},$$
$$\tau=\left(\frac{EI}{m+M}\right)^{1/2}\frac{t}{L^2},\ \omega=\left(\frac{m+M}{EI}\right)^{1/2}L^2\Omega,$$
$$u=\left(\frac{M}{EI}\right)^{1/2}UL,\ \beta=\frac{M}{m+M},\ \Gamma_s=\frac{m_s}{(m+M)L},$$
$$\Gamma_f=\frac{m_f}{(m+M)L}, \tag{2}$$

the equation of motion of the measuring tube (Eq. (1a)) and the corresponding boundary conditions (Eq. (1b)) can be written in the form

$$\frac{\partial^4\eta}{\partial\xi^4}+u^2\frac{\partial^2\eta}{\partial\xi^2}+2u\sqrt{\beta}\frac{\partial^2\eta}{\partial\xi\partial\tau}+\frac{\partial^2\eta}{\partial\tau^2}$$
$$+\Gamma_s\left[\delta\left(\xi-\xi_{s1}\right)+\delta\left(\xi-\xi_{s2}\right)\right]\frac{\partial^2\eta}{\partial\tau^2} \tag{3a}$$
$$+\Gamma_f\delta\left(\xi-\xi_f\right)\frac{\partial^2\eta}{\partial\tau^2}=f\delta\left(\xi-\xi_f\right)\cos(\omega\tau),$$

$$\eta\big|_{\xi=0}=0,\ \eta\big|_{\xi=1}=0,\ \left.\frac{\partial\eta}{\partial\xi}\right|_{\xi=0}=0,\ \left.\frac{\partial\eta}{\partial\xi}\right|_{\xi=1}=0. \tag{3b}$$

Applying the assumption of small flow-related terms, point masses, and forcing amplitude to Eq. (3a) yields

$$\frac{\partial^4\eta}{\partial\xi^4}+\frac{\partial^2\eta}{\partial\tau^2}+\varepsilon\left\{u^2\frac{\partial^2\eta}{\partial\xi^2}+2u\sqrt{\beta}\frac{\partial^2\eta}{\partial\xi\partial\tau}\right.$$

$$+\Gamma_s\left(\delta\left(\xi-\xi_{s1}\right)+\delta\left(\xi-\xi_{s2}\right)\right)\frac{\partial^2\eta}{\partial\tau^2} \tag{4a}$$

$$\left.+\Gamma_f\delta\left(\xi-\xi_f\right)\frac{\partial^2\eta}{\partial\tau^2}\right\}=\varepsilon f\delta\left(\xi-\xi_f\right)\cos(\omega\tau),$$

$$\eta\big|_{\xi=0}=0,\ \eta\big|_{\xi=1}=0,\ \left.\frac{\partial\eta}{\partial\xi}\right|_{\xi=0}=0,\ \left.\frac{\partial\eta}{\partial\xi}\right|_{\xi=1}=0. \tag{4b}$$

Equation (4a) is a typical continuous gyroscopic system with weak excitation, flow- and point-mass-related terms. In other words, ε as a bookkeeping parameter was used in order to show the terms assumed to be small compared to the other ones.

3. APPROXIMATE ANALYTIC SOLUTION

The aim of this section is to apply a perturbation analysis (the method of multiple scales) to the simplified mathematical model of a Coriolis flowmeter (Eq.4 (a,b)) in order to obtain an approximate closed-form solution for the phase of vibration response at each point of the measuring tube. This helps us to assess how various factors such as flow velocity, and point masses (representative of the sensors) can possibly influence the phase-shift between two points symmetrically located around the mid-tube of the measuring tube.

In the method of multiple scales [11-15], an approximation is sought in the form of following expansion:

$$\eta\left(\xi,\tau;\varepsilon\right)=\eta_0\left(\xi,T_0,T_1\right)+\varepsilon\eta_0\left(\xi,T_0,T_1\right)+O\left(\varepsilon^2\right), \tag{5}$$

where $T_0=\tau$ and $T_1=\varepsilon\tau$ are fast and slow timescales, respectively, and $\varepsilon<<1$; $O\left(\varepsilon^2\right)$ denotes terms of order of magnitude ε^2 and smaller.

The chain rule in time differentiation provides the relations

$$\frac{\partial}{\partial\tau}=\frac{\partial}{\partial T_0}+\varepsilon\frac{\partial}{\partial T_1}, \tag{6a}$$

$$\frac{\partial^2}{\partial\tau^2}=\frac{\partial^2}{\partial T_0^2}+2\varepsilon\frac{\partial^2}{\partial T\partial T_1}+O(\varepsilon^2). \tag{6b}$$

Substituting Eq. (5) into Eq. 4(a,b) and using Eq.6(a,b), as well as equating coefficients of like powers of ε, the equations of order 1 and ε and the corresponding boundary conditions take the form

$$O(\varepsilon^0):\frac{\partial^4\eta_0}{\partial\xi^4}+\frac{\partial^2\eta_0}{\partial T_0^2}=0, \tag{7a}$$

$$\eta_0\big|_{\xi=0}=0,\ \eta_0\big|_{\xi=1}=0,\ \left.\frac{\partial\eta_0}{\partial\xi}\right|_{\xi=0}=0,\ \left.\frac{\partial\eta_0}{\partial\xi}\right|_{\xi=1}=0. \tag{7b}$$

$$O(\varepsilon^1):\frac{\partial^4\eta_1}{\partial\xi^4}+\frac{\partial^2\eta_1}{\partial T_0^2}=-2\frac{\partial^2\eta_0}{\partial T_0\partial T_1}-\left\{u^2\frac{\partial^2\eta_0}{\partial\xi^2}\right.$$

$$+2u\sqrt{\beta}\frac{\partial^2\eta_0}{\partial\xi\partial T_0}+\Gamma_s\left(\delta\left(\xi-\xi_{s1}\right)+\delta\left(\xi-\xi_{s2}\right)\right)\frac{\partial^2\eta_0}{\partial T_0^2} \tag{8a}$$

$$\left.+\Gamma_f\delta\left(\xi-\xi_f\right)\frac{\partial^2\eta_0}{\partial T_0^2}\right\}+f\delta\left(\xi-\xi_f\right)\cos(\omega\tau),$$

$$\eta_1\big|_{\xi=0}=0,\ \eta_1\big|_{\xi=1}=0,\ \left.\frac{\partial\eta_1}{\partial\xi}\right|_{\xi=0}=0,\ \left.\frac{\partial\eta_1}{\partial\xi}\right|_{\xi=1}=0. \tag{8b}$$

The general solution for the equation order 1 (Eq. (7a)) may be expressed as a series expansion in terms of the slow timescale and complex-valued amplitude, $A_n\left(T_1\right)$, and a mode shape function $y_n(\xi)$, i.e.

$$\eta_0\left(\xi,T_0,T_1\right)=\sum_{n=1}^{\infty}\left(A_n\left(T_1\right)e^{i\omega_nT_0}+\overline{A}_n\left(T_1\right)e^{-i\omega_nT_0}\right)y_n(\xi), \tag{9}$$

where ω_n is the nth linear natural frequency of the unperturbed system ($\varepsilon=0$) and the overbar denotes the complex conjugate of the terms.

Substituting Eq.(9) into Eq.7(a,b) results in

$$\frac{d^4y_n}{d\xi^4}-\omega_n^2y_n=0, \tag{10a}$$

$$y_n\big|_{\xi=0}=0,\ y_n\big|_{\xi=1}=0,\ \left.\frac{dy_n}{d\xi}\right|_{\xi=0}=0,\ \left.\frac{dy_n}{d\xi}\right|_{\xi=1}=0. \tag{10b}$$

The solution of Eq.(10a) can be expressed as

$$\begin{aligned}y_n(\xi)=c_{1n}\Big[&\cos(\sqrt{\omega_n}\xi)+c_{2n}\sin(\sqrt{\omega_n}\xi)\\&+c_{3n}\cosh(\sqrt{\omega_n}\xi)+c_{4n}\sinh(\sqrt{\omega_n}\xi)\Big],\end{aligned} \tag{11}$$

where c_{1n}, c_{2n}, c_{3n} and c_{4n} are different constants.
Substituting Eq.(11) into the equations of boundary conditions (Eq.(10b)) leads to

$$\left[M\right]_{4\times4}\left[c_{1n} \quad c_{2n} \quad c_{3n} \quad c_{4n}\right]^{\mathrm{T}} = \left[0\right]_{4\times1}, \tag{12}$$

where, $\left[M\right]_{4\times4}$ is called the coefficient matrix.

In order to have nontrivial solution for $\left[c_{1n} \quad c_{2n} \quad c_{3n} \quad c_{4n}\right]^{\mathrm{T}}$, the determinant of the coefficient matrix should be equal to zero, which leads to the following characteristic equation:

$$\cos(\sqrt{\omega_n})\cosh(\sqrt{\omega_n})-1=0. \tag{13}$$

Employing the elimination process in Eq.(12) yields the following mode shape function:

$$\begin{aligned} y_n(\xi) = c_{1n}\Big[&\cos(\sqrt{\omega_n}\xi)-\cosh(\sqrt{\omega_n}\xi) \\ &-\frac{\cos(\sqrt{\omega_n})-\cosh(\sqrt{\omega_n})}{\sin(\sqrt{\omega_n})-\sinh(\sqrt{\omega_n})}\times \\ &\left(\sin(\sqrt{\omega_n}\xi)-\sinh(\sqrt{\omega_n}\xi)\right)\Big]. \end{aligned} \tag{14}$$

Equation (9) is general solution for Eq.(7a), including internal resonances. However, since this article considers the system only under resonant excitation at the fundamental (primary) natural frequency, ω_1 --for the flowmetering application-- it is sufficient to retain only the first mode, i.e.

$$\eta_0\left(\xi,T_0,T_1\right)=\left(A_1\left(T_1\right)e^{i\omega_1T_0}+\overline{A}_1\left(T_1\right)e^{-i\omega_1T_0}\right)y_1(\xi). \tag{15}$$

Substituting Eq.(15) into Eq.(8a) and expressing the trigonometric functions in exponential form yield

$$\begin{aligned} O(\varepsilon^1):\frac{\partial^4\eta_1}{\partial\xi^4}+\frac{\partial^2\eta_1}{\partial T_0^2}=\Bigg[&-2i\omega_1\frac{dA_1}{dT_1}y_1 \\ &-u^2A_1\frac{d^2y_1}{d\xi^2}-2u\sqrt{\beta}i\omega_1A_1\frac{dy_1}{d\xi} \\ &+\Gamma_s\left(\delta\left(\xi-\xi_{s1}\right)+\delta\left(\xi-\xi_{s2}\right)\right)\omega_1^2A_1y_1 \\ &+\Gamma_f\delta\left(\xi-\xi_f\right)\omega_1^2A_1y_1\Bigg]e^{i\omega_1T_0} \\ &+\frac{1}{2}f\delta\left(\xi-\xi_f\right)e^{i\omega\tau}+cc, \end{aligned} \tag{16}$$

where cc denotes the complex conjugate of all preceding terms.
Employing the standard procedure to study the primary resonance, which occurs when the excitation frequency, ω, is near the first natural frequency of the unperturbed system ($\varepsilon=0$), a detuning parameter, σ, is introduced such that

$$\omega=\omega_1+\varepsilon\sigma\ . \tag{17}$$

Equation (16) is an inhomogeneous partial differential equation which should be solved for η_1. Assuming a particular solution of η_1 in the form

$$\eta_1\left(\xi,T_0,T_1\right)=\left[\sum_{j=2}^{k}\left(X_j\left(T_1\right)y_j(\xi)\right)\right]e^{i\omega_1T_0}+cc, \tag{18}$$

and inserting it into Eq.(16) results in

$$\begin{aligned} &-\omega_1^2\left(\sum_{j=2}^{k}\left(X_j\left(T_1\right)y_j(\xi)\right)\right)+\sum_{j=2}^{k}\left(X_j\left(T_1\right)\frac{d^4y_j}{d\xi^4}\right) \\ &=\Bigg[-2i\omega_1\frac{dA_1}{dT_1}y_1-u^2A_1\frac{d^2y_1}{d\xi^2}-2u\sqrt{\beta}i\omega_1A_1\frac{dy_1}{d\xi} \\ &+\Gamma_s\left(\delta\left(\xi-\xi_{s1}\right)+\delta\left(\xi-\xi_{s2}\right)\right)\omega_1^2A_1y_1 \\ &+\Gamma_f\delta\left(\xi-\xi_f\right)\omega_1^2A_1y_1\Bigg]e^{i\omega_1T_0} \\ &+\frac{1}{2}f\delta\left(\xi-\xi_f\right)e^{i\omega\tau}+cc. \end{aligned} \tag{19}$$

Multiplying both sides of Eq.(19) by any $y_s(\xi)$, $s=1,2,3,...,n$, and integrating over ξ yields

$$\begin{aligned} X_j\left[\left(\frac{\omega_j}{\omega_1}\right)-1\right]=\omega_1^{-2}\int_0^1\Bigg(&-2i\omega_1\frac{dA_1}{dT_1}y_1 \\ &-u^2A_1\frac{d^2y_1}{d\xi^2}-2u\sqrt{\beta}i\omega_1A_1\frac{dy_1}{d\xi} \\ &+\Gamma_s\left(\delta\left(\xi-\xi_{s1}\right)+\delta\left(\xi-\xi_{s2}\right)\right)\omega_1^2A_1y_1 \end{aligned}$$

$$+\Gamma_f \delta\left(\xi-\xi_f\right)\omega_1^2 A_1 y_1 \Bigg) y_j d\xi \tag{20}$$

$$+\frac{1}{2}\omega_1^{-2} f e^{i\sigma T_1} y_j(\xi=\xi_f), \qquad j=1,2,3,....,n.$$

For $j=1$ and 2 in Eq.(20), the following equations can be obtained:

$$\int_0^1 \Bigg(-2i\omega_1 \frac{dA_1}{dT_1} y_1 - u^2 A_1 \frac{d^2 y_1}{d\xi^2} - 2u\sqrt{\beta} i\omega_1 A_1 \frac{dy_1}{d\xi}$$
$$+\Gamma_s\left(\delta\left(\xi-\xi_{s1}\right)+\delta\left(\xi-\xi_{s2}\right)\right)\omega_1^2 A_1 y_1$$
$$+\Gamma_f \delta\left(\xi-\xi_f\right)\omega_1^2 A_1 y_1 \Bigg) y_j d\xi \tag{21a}$$
$$+\frac{1}{2} f e^{i\sigma T_1} y_j(\xi=\xi_f)=0,$$

$$X_2\left[\omega_2^2-\omega_1^2\right]=\int_0^1 \Bigg(-2i\omega_1 \frac{dA_1}{dT_1} y_1$$
$$-u^2 A_1 \frac{d^2 y_1}{d\xi^2} - 2u\sqrt{\beta} i\omega_1 A_1 \frac{dy_1}{d\xi}$$
$$+\Gamma_s\left(\delta\left(\xi-\xi_{s1}\right)+\delta\left(\xi-\xi_{s2}\right)\right)\omega_1^2 A_1 y_1 \tag{21b}$$
$$+\Gamma_f \delta\left(\xi-\xi_f\right)\omega_1^2 A_1 y_1 \Bigg) y_2 d\xi$$
$$+\frac{1}{2}\omega_1^{-2} f e^{i\sigma T_1} y_2(\xi=\xi_f).$$

Calculating the integrals associated with the mode shape function in the *solvability condition* (Eq.(21a)) and simplifying the resultant equation gives

$$\frac{dA_1}{dT_1}+\alpha_1 A_1+\frac{1}{2}\alpha_2 e^{i\sigma T_1}=0\,, \tag{22}$$

Where

$$\alpha_1=\frac{-u^2\int_0^1 y_1 \frac{d^2 y_1}{d\xi^2} d\xi - 2u\sqrt{\beta} i\omega_1 \int_0^1 y_1 \frac{dy_1}{d\xi} d\xi}{-2i\omega_1\int_0^1 y_1^2 d\xi}$$
$$+\frac{\Gamma_s\omega_1^2\left(y_1^2(\xi=\xi_{s1})+y_1^2(\xi=\xi_{s2})\right)}{-2i\omega_1\int_0^1 y_1^2 d\xi} \tag{23a}$$
$$+\frac{\Gamma_f\omega_1^2 y_1^2(\xi=\xi_f)}{-2i\omega_1\int_0^1 y_1^2 d\xi},$$

$$\alpha_2=\frac{f y_1(\xi=\xi_f)}{-2i\omega_1\int_0^1 y_1^2 d\xi}. \tag{23b}$$

Since $\int_0^1 y_1\,(dy_1/d\xi)d\xi=0$, Eq.23(a,b) shows that α_1 and α_2 are pure imaginary values. Then one may write

$$\alpha_1=i\alpha_1^i \quad \text{and} \quad \alpha_2=i\alpha_2^i. \tag{24}$$

In order to solve Eq.(22), which is a linear first-order ordinary differential equation, one may express the slow timescale dependent amplitude in polar form; i.e.

$$A_1\left(T_1\right)=\frac{1}{2}a_1\left(T_1\right)e^{i\beta_1(T_1)}, \tag{25}$$

where $a_1\left(T_1\right)$ and $\beta_1\left(T_1\right)$ represent the real-valued, slow timescale dependent amplitude and phase of the response, respectively.
Inserting Eq.(25) into Eq.(22), using Eq.(24), and separating the resultant equation into real and imaginary components gives

$$\frac{da_1}{dT_1}-\alpha_2^i\sin(\gamma)=0, \tag{26a}$$

$$a_1\left(\sigma-\frac{d\gamma}{dT_1}\right)+a_1\alpha_1^i-\alpha_2^i\cos(\gamma)=0, \tag{26b}$$

where $\gamma=\sigma T_1-\beta$. Equation 26 (a,b) governs the slowly varying amplitude and phase corresponding to the first mode. If the slow-timescale-dependent amplitude and phase (a_1 and β_1) do not change with time, the steady-state response (limit cycle) are obtained. Fulfilling this condition in Eq.26 (a,b) and solving the resulting equations for a_1 and γ result in

$$\gamma=0 \text{ (or } \beta=\sigma T_1),\;\; a_1=-\frac{\alpha_2^i}{\sigma+\alpha_1^i}. \tag{27}$$

Inserting Eq.27 into Eq.(25), $A_1(T_1)$ becomes

$$A_1(T_1) = -\frac{\alpha_2^i}{2\left(\sigma+\alpha_1^i\right)} e^{i\sigma T_1}. \tag{28}$$

In Eq.(28), $A_1(T_1)$ has been determined and in what follows, the aim is to determine $X_2(T_1)$. For this, considering Eq. (21b), calculating the integrals and simplifying the resulting equation yield

$$X_2(T_1) = \alpha_3 A_1 + \alpha_4 e^{i\sigma T_1}, \tag{29a}$$

where

$$\begin{aligned}\alpha_3 = \frac{1}{\omega_2^2-\omega_1^2}\Bigg(&-u^2\int_0^1 y_2 \frac{d^2 y_1}{d\xi^2} d\xi \\ &-2u\sqrt{\beta} i\omega_1 \int_0^1 y_2 \frac{dy_1}{d\xi} d\xi \\ &+\Gamma_s \omega_1^2 \left(y_1(\xi=\xi_{s1}) y_2(\xi=\xi_{s1}) \right. \\ &\left. + y_1(\xi=\xi_{s2}) y_2(\xi=\xi_{s2}) \right) \\ &+\Gamma_f \omega_1^2 y_1(\xi=\xi_f) y_2(\xi=\xi_f) \Bigg),\end{aligned} \tag{29b}$$

$$\alpha_4 = \frac{f y_2(\xi=\xi_f)}{\left(\omega_2^2-\omega_1^2\right)} = \alpha_4^r; \tag{29c}$$

in view of Eq.29 (b,c), α_3 and α_4 are complex- and real-valued quantities, respectively.

Substituting $A_1(T_1)$ from Eq.(28) into Eq.(29a), $X_2(T_1)$ can be determined as

$$X_2(T_1) = \left(-\frac{\alpha_2^i(\alpha_3^r + i\alpha_3^i)}{2(\sigma+\alpha_1^i)} + \alpha_4^r \right) e^{i\sigma T_1}. \tag{30}$$

Using Eqs.(15) and (28), η_0 becomes

$$\eta_0(\xi, T_0, T_1) = -\frac{\alpha_2^i}{\sigma+\alpha_1^i} \cos(\omega\tau) y_1(\xi). \tag{31}$$

Considering Eq.(18) along with Eqs.(28) and (30), the solution of equation of order ε takes the form

$$\begin{aligned}\eta_1(\xi, T_0, T_1) = \Bigg[&\left(-\frac{\alpha_2^i \alpha_3^r}{\sigma+\alpha_1^i} + 2\alpha_4^r \right)\cos(\omega\tau) \\ &+ \frac{\alpha_2^i \alpha_3^i}{\sigma+\alpha_1^i}\sin(\omega\tau)\Bigg] y_2(\xi).\end{aligned} \tag{32}$$

Using Eqs. (5), (31) and (32), the two-mode approximate solution for the measuring tube response becomes

$$\begin{aligned}\eta(\xi,\tau) = &-\frac{\alpha_2^i}{\sigma+\alpha_1^i}\cos(\omega\tau) y_1(\xi) \\ &+\varepsilon\Bigg[\left(-\frac{\alpha_2^i \alpha_3^r}{\sigma+\alpha_1^i} + 2\alpha_4^r \right)\cos(\omega\tau) \\ &+ \frac{\alpha_2^i \alpha_3^i}{\sigma+\alpha_1^i}\sin(\omega\tau)\Bigg] y_2(\xi) + O\left(\varepsilon^2\right).\end{aligned} \tag{33}$$

As ε approaches zero in Eq.(33), the measuring tube vibrates in its driven fundamental mode $y_1(\xi)$ with forcing frequency ω. When $\varepsilon \neq 0$, there is an additional superimposed motion to the response of the unperturbed system, η_0; this motion is the effects of the second mode, $y_2(\xi)$, on the response.

Rearranging the terms of Eq.(33) gives

$$\eta(\xi,\tau) = G_1(\xi)\cos(\omega\tau) - G_2(\xi)\sin(\omega\tau), \tag{34a}$$

where

$$G_1(\xi) = -\frac{\alpha_2^i}{\sigma+\alpha_1^i} y_1(\xi) + \varepsilon\left(-\frac{\alpha_2^i \alpha_3^r}{\sigma+\alpha_1^i} + 2\alpha_4^r \right) y_2(\xi), \tag{34b}$$

$$G_1(\xi) = -\varepsilon \frac{\alpha_2^i \alpha_3^i}{\sigma+\alpha_1^i} y_2(\xi). \tag{34c}$$

It can be concluded from Eq. (34a) that the second term of the response $\eta(\xi,\tau)$ is phase-shifted by 90 degrees in time with respect to the first one. Equation (33) illustrates that the measuring tube vibrates in the first mode, $y_1(\xi)$ --which is symmetric-- along with a small motion (because multiplied by ε) resulting from the antisymmetric second mode, $y_2(\xi)$. The phase difference between these motions is 90 degrees; hence, the resulting motion is of a traveling-wave type.

Simplifying Eq.(34) using trigonometric identities results in

$$\eta(\xi,\tau) = \sqrt{(G_1(\xi))^2 + (G_2(\xi))^2}\cos(\omega\tau + \psi(\xi)), \tag{35a}$$

where

$$\psi(\xi) = \arctan\left(\frac{G_2(\xi)}{G_1(\xi)}\right), \tag{35b}$$

and $\psi(\xi)$ is the ξ-dependent phase of the response at each point of the measuring tube.

4. NUMERICAL EXAMPLES

In this section we present numerical results for the phase-shift along the measuring tube. Moreover, the effects of system parameters such as added masses, mass parameter β, and sensor locations are discussed.

In the calculations, the following parameters have been used: L=0.2032 m, r=5.08 mm, h=0.508 mm, E=99.974 × 10^9 N/m^2, ρ_p=4480 kg/m^3, ρ_f=1000 kg/m^3, and m_s= m_f=0.0047 kg. In Coriolis flowmetering applications, the case of common interest is for ξ_f=0.5; i.e. the external periodic excitation is usually applied at mid-tube and a sharp fundamental resonance is maintained using a feedback control. Therefore, in this section ξ_f=0.5 is considered for all numerical analysis.

The dependence of the phase of vibration response of the measuring tube on the dimensionless axial coordinate, ξ, is shown in Fig.2. As ξ increases, the phase increases from a negative value to positive one.

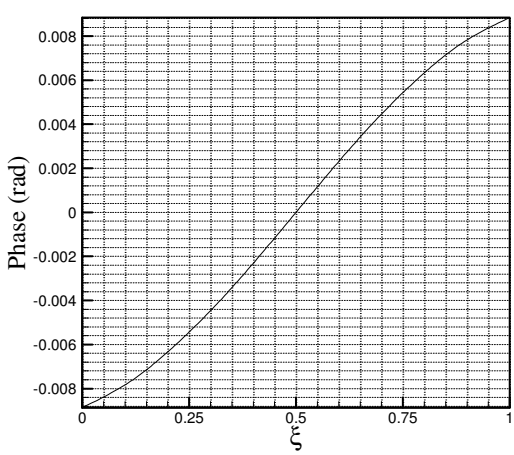

Figure 2: The phase of vibration response of the measuring tube as a function of dimensionless axial coordinate;

$u = 0.1,\ \beta = 0.5152,\ \Gamma_s = 0.147,\ \Gamma_f = 0.147,$
$f = 0.001,\ \xi_{s1} = 0.25,\ \xi_{s2} = 0.75,\ \xi_f = 0.5$

The phase-shift between two symmetrically located points of the measuring tube around mid-tube at ξ_{s1}=0.25 and ξ_{s1}=0.75 versus the dimensionless flow velocity for a selection of mass parameter values is shown in Fig.3. It is seen that the phase-shift is linearly proportional to u and the factor of proportionality depends on β. This means, in the application, once the phase-shift is measured from time traces recorded from sensors, $G_1(\xi)$ and $G_2(\xi)$ can be determined in Eq.34(a,b). Then using Eqs.23(a,b) and 29(b,c), for given values of β, Γ_s, Γ_s and f, the dimensionless flow velocity u can be determined.

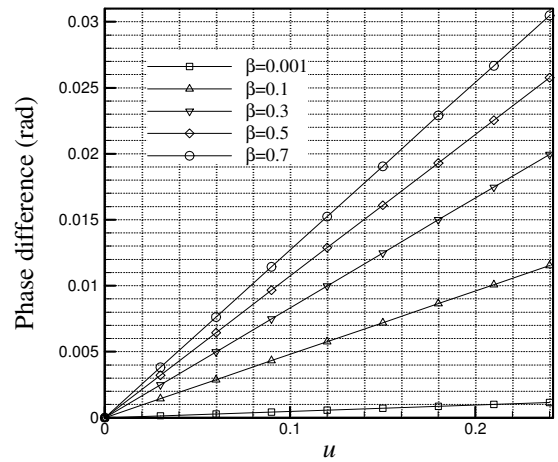

Figure 3: The phase-shift of vibration response of the measuring tube as a function of dimensionless flow velocity;

$\Gamma_s = 0.147,\ \Gamma_f = 0.147,\ f = 0.001,\ \xi_{s1} = 0.25,\ \xi_{s2} = 0.75,\ \xi_f = 0.4.$

The influence of the location of the sensors (closeness to or being far from the mid-tube) on the phase-shift is shown in Fig.4 for a system that the response is measured at two points symmetrically located around the mid-tube, i.e. $\xi_{s2} - 0.5 = 0.5 - \xi_{s1}$. As seen here, the farther the two sensors are located from mid-tube, the larger phase-shift is predicted.

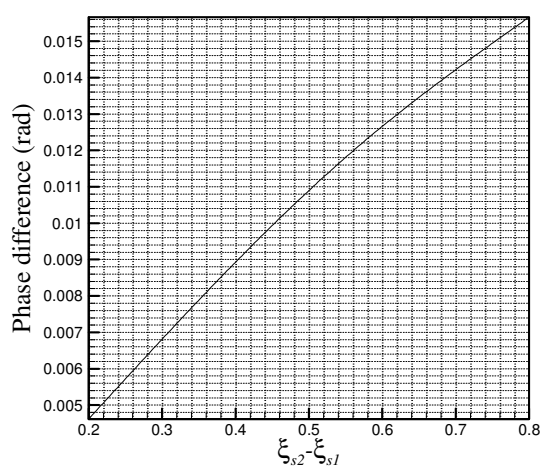

Figure 4: The phase-shift of vibration response of the measuring tube as a function of sensors location;

$u = 0.1,\ \beta = 0.5152,\ \Gamma_s = 0.147,\ \Gamma_f = 0.147,\ f = 0.01,\ \xi_f = 0.5.$

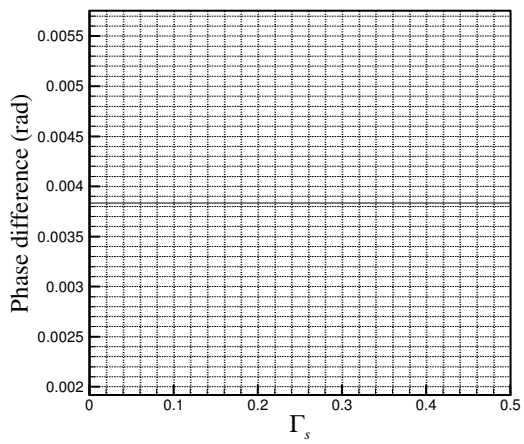

Figure 5: The phase of vibration response of the measuring tube as a function of point-mass parameter Γ_s; $u = 0.08,\ \beta = 0.1,\ \Gamma_f = 0.147,\ f = 0.01,\ \xi_{s1} = 0.25,\ \xi_{s2} = 0.75,\ \xi_f = 0.5.$

Figure 5 shows the variation in phase-shift with Γ_s. Surprisingly, Γ_s does not affect the phase-shift, to the order of approximation used in this analysis. In fact, numerical results show that the effect of Γ_s is very small and of cubic or smaller order than other terms; hence, Γ_s effect is negligible.

5. SUMMARY AND CONCLUSIONS

In this paper, the phase-shift along the length of a measuring tube of a Coriolis flowmeter is determined analytically by means of the method of multiple scales. The closed-form expression for spatial coordinate-related phase-shift provides direct insight into which parameters influence the proportionality between the phase-shift and flow velocity (thus mass flow).

Numerical results show that increasing the flow velocity and mass parameter β increase the phase-shift; for higher values of β, the factor of proportionality between flow velocity and phase-shift is larger. Similarly, this proportionality is large for the large distances between the sensors. On the other hand, although the point-mass parameter Γ_s affects the vibration response of the measuring tube, it has no influence on the phase-shift, to the order of approximation employed here. This implies that the effect of Γ_s is smaller than that of the other parameters of the system.

In conclusion, it should be noted that the analytical expression for phase-shift presented in this article is for a simple model of single, straight, measuring tube with added masses conveying uniform plug flow. However, much accurate results may be predicted by considering the effects of some other parameters, e.g. external vibrations due to pumping system, neighbouring vibrating devices, flow compressibility and flexible supports.

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