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# PARAMETER ESTIMATION OF A FLUTTERING AEROELASTIC SYSTEM IN THE TRANSITIONAL REYNOLDS NUMBER REGIME

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## ABSTRACT

*We report the parameter estimation results of a self-sustaining aeroelastic oscillator. The system is composed of a rigid wing that is elastically mounted on a rig, which in turn is fixed in a wind tunnel. For certain flow conditions, in particular dictated by the Reynolds number in the transitional regime, the wing extracts energy from the flow leading to a stable limit cycle oscillation. The basic physical mechanism at the origin of the oscillations is laminar boundary layer separation, which leads to negative aerodynamic damping. An empirical model of the aeroelastic system is proposed in the form of a generalized Duffing-van der Pol oscillator, whereby the linear and nonlinear aeroelastic terms are unknowns to be estimated. The model (input) noise process accounting for the amplitude modulation observed from experiments will also be estimated. We apply a Bayesian inference based batch data assimilation method in tackling this strongly nonlinear and non-Gaussian model. In particular, Markov Chain Monte Carlo sampling technique is used to generate samples from the joint distribution of the unknown parameters given noisy measurement data. The extended Kalman filter is utilized to obtain the conditional distribution of the model state given the noisy measurements. The parameter estimates for a third order generalized Duffing-van der Pol oscillator are obtained and marginal and joint probability density functions for the parameters will be presented for both a numerical model and a rigid wing that is elastically mounted on a rig in a wind tunnel.*

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## 1 INTRODUCTION

The use of flight vehicles operating in low-Reynolds-number regime has recently increased in both military and civil sectors. In this range of Reynolds numbers, i.e. $10^4 \leq Re \leq 10^6$, laminar boundary layer separation may occur leading to the formation of a laminar separation bubble (LSB), transition of the laminar shear layer, and subsequent re-attachment of the turbulent layer [1]. Experimental aero-elastic investigations involving a freely-rotating NACA 0012 airfoil in this Reynolds-number regime showed self-sustained oscillations of the airfoil[2, 3]. The physical mechanism causing the self-sustained oscillations is laminar boundary layer separation leading to negative aerodynamic damping. In previous work by the authors, an empirical model of the aeroelastic system was proposed in the form of a generalized Duffing-van der Pol oscillator[4], whereby the linear and nonlinear aeroelastic terms were then estimated from joint state and parameter estimation using ensemble Kalman filtering technique[5].

In this paper, we apply Bayesian inference parameter estimation technique to estimate these parameters for a numerical model. Furthermore, the model (input) noise process accounting for the amplitude modulation observed from experiments will also be estimated. To achieve this, Markov Chain Monte Carlo (MCMC) sampling technique [6, 7] is used to generate samples from the joint distribution of the unknown parameters given noisy measurement data. The samples obtained from MCMC runs provide estimated marginal and joint densities of the parameters to be estimated. The evaluation of the joint probability density function (pdf) of these parameters requires the evaluation of the conditional distribution of the model state (pitch and angular velocity) given the noisy measurements. The state estimation

 

is carried out using the extended Kalman filter (EKF) [8–10], which provides a Gaussian joint pdf approximation of the system state state. The assimilation of dense measurements in time renders the system state weakly non-Gaussian for a nonlinear system and thus EKF is well suited as a state estimation technique here. The parameter estimates obtained using the proposed technique will be compared with those obtained using Least-Squares estimation.

## 2 THEORY

Bayesian inference provides a statistical framework which relates physical observations to mathematical models[6, 7, 11]. Let $\theta$ be the vector of system parameters and the vector $\mathbf{d}$ denotes a collection of physical observations. In Bayesian settings, the conditional pdf $p(\mathbf{d}|\theta)$ of the observations given the system parameters is called the *forward* pdf or likelihood function. When observations are available, Bayesian inference deduces the value of the system parameters relying on a valid mathematical model. This step provides the conditional pdf $p(\theta|\mathbf{d})$ of the system parameters given the observations, denoted by the *inverse* or posterior pdf. Bayes theorem relates the inverse and forward pdf[6, 7, 11]:

$$p(\theta|\mathbf{d}) = \frac{p(\mathbf{d}|\theta)\,p(\theta)}{p(\mathbf{d})}. \tag{1}$$

In Eq. (1), $p(\theta)$ is the pdf that contains the available knowledge of the parameters before making the observation $\mathbf{d}$[6, 7, 11]. It is called the *prior* pdf of $\theta$. On the left-hand side of Eq. (1), we have the *posterior* pdf $p(\theta|\mathbf{d})$ of $\theta$ that represents our knowledge of the system parameters after making the observations. To obtain the posterior pdf, the prior pdf is multiplied by the forward or *likelihood* pdf $p(\mathbf{d}|\theta)$ which can be characterized by the mathematical model of the system. The factor $p(\mathbf{d}) = \int p(\mathbf{d}|\theta)\,p(\theta)\,\mathrm{d}\theta$ is just a normalization constant.

### 2.1 State Space Models

The discrete state-space representation of a nonlinear system is given by [8–10, 12–17]

$$\mathbf{x}_{k+1} = \mathbf{g}_k(\mathbf{x}_k, \mathbf{f}_k, \mathbf{q}_k). \tag{2}$$

Here $\mathbf{x} \in \mathbb{R}^n$ is the state vector, $\mathbf{g} \in \mathbb{R}^n$ is the discrete nonlinear model operator, $\mathbf{f} \in \mathbb{R}^p$ is a deterministic input and $\mathbf{d} \in \mathbb{R}^m$ is the measurement vector which relates to the true state by the measurement operator $\mathbf{h} \in \mathbb{R}^m$. $\mathbf{q} \in \mathbb{R}^s$ and $\varepsilon \in \mathbb{R}^r$ are independent Gaussian random vectors with mean $\overline{\mathbf{q}_k} \in \mathbb{R}^s$ and $\overline{\varepsilon_k} \in \mathbb{R}^r$ and covariance matrices $\mathbf{Q} \in \mathbb{R}^{s\times s}$ and $\Gamma \in \mathbb{R}^{r\times r}$ respectively. We assume that the model depends on some unknown parameter vector $\theta$.

The state vector $\mathbf{x}$ is a hidden Markov process with a transition density $p(\mathbf{x}_k|\mathbf{x}_{k-1})$. One can obtain observational data of this process modelled by the following measurement equation

$$\mathbf{d}_k = \mathbf{h}_k(\mathbf{x}_k, \varepsilon_k) \tag{3}$$

where $\mathbf{d}_k$ is the measurement vector which relates to the true state $\mathbf{x}_k$ and measurement noise $\varepsilon_k$ by the measurement operator $\mathbf{h}_k$. From this observational model one can also obtain a conditional density of the measurements given the state, denoted by $p(\mathbf{d}_k|\mathbf{x}_k)$.

We can obtain the posterior pdf of the unknown parameter vector given the measurements as follows[18]

$$\begin{aligned} p(\theta|\mathbf{d}_{1:n}) &\propto p(\mathbf{d}_{1:n}|\theta)\,p(\theta) \\ &= p(\theta)\prod_{k=1}^{n} p(\mathbf{d}_k|\theta) \\ &= p(\theta)\prod_{k=1}^{n}\int p(\mathbf{d}_k|\mathbf{x}_k)\,p(\mathbf{x}_k|\theta,\mathbf{d}_{1:k-1})\,d\mathbf{x}_k. \end{aligned} \tag{4}$$

We need to obtain $p(\mathbf{x}_k|\theta,\mathbf{d}_{1:k-1})$ being the posterior density of the state $\mathbf{x}_k$ at time $t_k$ given the observations $\mathbf{d}_{1:k-1}$ at time steps up to but not including $t_k$ and the unknown parameter vector $\theta$. We resort to filtering techniques to obtain this conditional (or posterior) distribution.

### 2.2 Data Assimilation via Filtering

The Kalman filter (KF) has become a popular tool for state estimation problems in linear systems (i.e. linear model and measurement operators) due to its mathematical simplicity [19, 20]. KF results in a recursive analytical solution of the *posterior* distribution of the system state conditional upon measurement data. For weakly nonlinear systems, KF can be extended to the nonlinear case by linearising the model and measurement operators around the current estimate of the state vector, leading to the popular (but no longer optimal) extended Kalman filter (EKF) [9, 10]. Both KF and EKF provide a Gaussian posterior pdf of the state given observational data. We will briefly overview EKF as a tool to obtain $p(\mathbf{x}_k|\theta,\mathbf{d}_{1:k-1})$ which would allow us to evaluate the posterior pdf in Eq. (4) of the unknown parameters.

#### 2.2.1 Extended Kalman Filter

For nonlinear systems, the pdf of the state vector $\mathbf{x}$ is generally non-Gaussian even if the model noise is additive and Gaussian. For weakly non-Gaussian behavior, one can reasonably approximate the conditional pdf of $\mathbf{x}$ by a Gaussian process through linearization.

One can linearize the measurement operator $\mathbf{h}_k$ in Eq. (3) about $\mathbf{x}_k = \mathbf{x}_k^f$ and $\varepsilon_k = \overline{\varepsilon_k}$ to obtain a linearized measurement model given by

$$\begin{aligned} \mathbf{h}_k(\mathbf{x}_k,\varepsilon_k) \approx\ & \mathbf{h}_k\left(\mathbf{x}_k^f, \overline{\varepsilon_k}\right) + \left.\frac{\partial \mathbf{h}_k(\mathbf{x}_k,\varepsilon_k)}{\partial \mathbf{x}_k}\right|_{\mathbf{x}_k=\mathbf{x}_k^f,\varepsilon_k=\overline{\varepsilon_k}}\left(\mathbf{x}_k - \mathbf{x}_k^f\right) \\ & + \left.\frac{\partial \mathbf{h}_k(\mathbf{x}_k,\varepsilon_k)}{\partial \varepsilon_k}\right|_{\mathbf{x}_k=\mathbf{x}_k^f,\varepsilon_k=\overline{\varepsilon_k}}(\varepsilon_k - \overline{\varepsilon_k}) \end{aligned} \tag{5}$$

where $\frac{\partial \mathbf{h}_k(\mathbf{x}_k,\varepsilon_k)}{\partial \mathbf{x}_k} \in \mathbb{R}^{m\times n}$ and $\frac{\partial \mathbf{h}_k(\mathbf{x}_k,\varepsilon_k)}{\partial \varepsilon_k} \in \mathbb{R}^{m\times r}$ describe the Jacobian matrices of $\mathbf{h}_k(\mathbf{x}_k,\varepsilon_k)$ with respect to $\mathbf{x}_k$ and $\varepsilon_k$ respectively. It is further assumed that $\mathbf{x}$ has a Gaussian prior pdf given by $\mathbf{x}_k \sim \mathcal{N}\left(\mathbf{x}_k^f, \mathbf{P}_k^f\right)$. In the analysis step, EKF estimates the conditional mean $\mathbf{x}_k^a$ and covariance $\mathbf{P}_k^a$ of $\mathbf{x}_k$ given the measurement vector $\mathbf{d}_k$:

Analysis step [8–10]:

$$\mathbf{C}_k = \left.\frac{\partial \mathbf{h}_k(\mathbf{x}_k,\varepsilon_k)}{\partial \mathbf{x}_k}\right|_{\mathbf{x}_k=\mathbf{x}_k^f,\varepsilon_k=\overline{\varepsilon_k}}, \tag{6}$$

$$\mathbf{D}_k = \left.\frac{\partial \mathbf{h}_k(\mathbf{x}_k,\varepsilon_k)}{\partial \varepsilon_k}\right|_{\mathbf{x}_k=\mathbf{x}_k^f,\varepsilon_k=\overline{\varepsilon_k}}, \tag{7}$$

$$\mathbf{K}_k = \mathbf{P}_k^f \mathbf{C}_k^T \left[\mathbf{D}_k \Gamma_k \mathbf{D}_k^T + \mathbf{C}_k \mathbf{P}_k^f \mathbf{C}_k^T\right]^{-1}, \tag{8}$$

$$\mathbf{x}_k^a = \mathbf{x}_k^f + \mathbf{K}_k\left(\mathbf{d}_k - \mathbf{h}_k\left(\mathbf{x}_k^f, \overline{\varepsilon_k}\right)\right), \tag{9}$$

$$\mathbf{P}_k^a = \left[\mathbf{I} - \mathbf{K}_k \mathbf{C}_k\right] \mathbf{P}_k^f. \tag{10}$$

We can also linearize the model operator $\mathbf{g}_k$ in Eq. (2) about $\mathbf{x}_k = \mathbf{x}_k^a$ and $\mathbf{q}_k = \overline{\mathbf{q}_k}$ to obtain a linearized model equation:

$$\begin{aligned}\mathbf{g}_k(\mathbf{x}_k,\mathbf{f}_k,\mathbf{q}_k) \approx\ & \mathbf{g}_k(\mathbf{x}_k^a,\mathbf{f}_k,\overline{\mathbf{q}_k}) \\ &+ \left.\frac{\partial \mathbf{g}_k(\mathbf{x}_k,\mathbf{f}_k,\mathbf{q}_k)}{\partial \mathbf{x}_k}\right|_{\mathbf{x}_k=\mathbf{x}_k^a,\mathbf{q}_k=\overline{\mathbf{q}_k}} (\mathbf{x}_k - \mathbf{x}_k^a) \\ &+ \left.\frac{\partial \mathbf{g}_k(\mathbf{x}_k,\mathbf{f}_k,\mathbf{q}_k)}{\partial \mathbf{q}_k}\right|_{\mathbf{x}_k=\mathbf{x}_k^a,\mathbf{q}_k=\overline{\mathbf{q}_k}} (\mathbf{q}_k - \overline{\mathbf{q}_k})\end{aligned} \tag{11}$$

where $\frac{\partial \mathbf{g}_k(\mathbf{x}_k,\mathbf{f}_k,\mathbf{q}_k)}{\partial \mathbf{x}_k} \in \mathbb{R}^{n\times n}$ and $\frac{\partial \mathbf{g}_k(\mathbf{x}_k,\mathbf{f}_k,\mathbf{q}_k)}{\partial \mathbf{q}_k} \in \mathbb{R}^{n\times s}$ describe the tangent linear model operators, being the Jacobian matrices of $\mathbf{g}_k(\mathbf{x}_k,\mathbf{f}_k,\mathbf{q}_k)$ with respect to $\mathbf{x}_k$ and $\mathbf{q}_k$ respectively.

From Eqs. (2)-(11), one obtains the posterior mean and error covariance matrix [8–10, 12–17] which constitutes the forecast step.

Forecast step [8–10]:

$$\mathbf{A}_k = \left.\frac{\partial \mathbf{g}_k(\mathbf{x}_k,\mathbf{f}_k,\mathbf{q}_k)}{\partial \mathbf{x}_k}\right|_{\mathbf{x}_k=\mathbf{x}_k^a,\mathbf{q}_k=\overline{\mathbf{q}_k}}, \tag{12}$$

$$\mathbf{B}_k = \left.\frac{\partial \mathbf{g}_k(\mathbf{x}_k,\mathbf{f}_k,\mathbf{q}_k)}{\partial \mathbf{q}_k}\right|_{\mathbf{x}_k=\mathbf{x}_k^a,\mathbf{q}_k=\overline{\mathbf{q}_k}}, \tag{13}$$

$$\mathbf{x}_{k+1}^f = \mathbf{g}_k(\mathbf{x}_k^a,\mathbf{f}_k,\overline{\mathbf{q}_k}), \tag{14}$$

$$\mathbf{P}_{k+1}^f = \mathbf{A}_k \mathbf{P}_k^a \mathbf{A}_k^T + \mathbf{B}_k \mathbf{Q}_k \mathbf{B}_k^T. \tag{15}$$

### 2.3 Markov Chain Monte Carlo Sampling

EKF can be applied to obtain the conditional pdf $p(\mathbf{x}_k|\theta,\mathbf{d}_{1:k-1})$ which would allow us to evaluate the posterior pdf in Eq. (4) of the unknown parameters. It is important to notice that the the normalization constant $p(\mathbf{d})$ is hard to evaluate. When the conditional pdf is only known up to a normalization, direct Monte Carlo sampling techniques are difficult to apply. One way to alleviate this difficulty is to use MCMC which does not require the knowledge of this constant.

Simple Monte Carlo sampling technique (e.g. [6]) can generate *independent* random samples from the posterior pdf $p(\theta|\mathbf{d})$ from which one can extract relevant statistical information (e.g. mean and mode). For a general non-Gaussian posterior pdf, independent samples are difficult to simulate in practice. One can however sample from a Markov chain whose *equilibrium* or *stationary* pdf matches the posterior pdf $p(\theta|\mathbf{d})$ as described in the following Subsection. Although the samples are not independent, the statistical features of the posterior pdf can be estimated with reasonable accuracy using a large enough sample size. In this Section, we briefly describe Markov chains and present the general algorithm of MCMC adopted in this report, namely the Metropolis algorithm. More details can be obtained in the books by Gilks *et al.* [6] and Liu [7].

#### 2.3.1 Markov Chains

A sequence of random vectors $\{\theta_0,\theta_1,\ldots\}$ is called a first-order Markov chain if it satisfies the property[7]

$$p(\theta_{i+1}=\mathbf{y}|\theta_i=\mathbf{z},\ldots,\theta_0=\mathbf{z}) = p(\theta_{i+1}=\mathbf{y}|\theta_i=\mathbf{z}). \tag{16}$$

It states that $\theta_{i+1}$ statistically depends only on the previous random vector $\theta_i$. When the transition pdf (kernel) $p(\theta_{i+1}=\mathbf{y}|\theta_i=\mathbf{z})$ is time-invariant (independent of $i$), it can be denoted by $A(\mathbf{z},\mathbf{y})$.

The pdf $p(\theta_i)$ converges to a stationary pdf $\pi(\theta_i)$ as $i\to\infty$ if the chain satisfies three properties[6]: (1) the chain is irreducible, i.e. starting with any possible state for $\theta_0$ the chain can reach any other state in some number of iterations; (2) the chain is aperiodic, i.e. the chain does not periodically oscillate among different sets of states; (3) the chain is positive recurrent, i.e. if the value of $\theta_0$ is sampled from the stationary distribution $\pi$, all subsequent iterates will be distributed according to $\pi$. The last condition is met if the stationary pdf $\pi$ and the transition kernel $A(\mathbf{z},\mathbf{y})$ satisfy the following eigenvalue problem:

$$\int \pi(\mathbf{z}) A(\mathbf{z},\mathbf{y})\,\mathrm{d}\mathbf{z} = \pi(\mathbf{y}). \tag{17}$$

The left-hand side of Eq. (17) gives the marginal distribution of $\mathbf{y}$ under the assumption that $\mathbf{z}$ is from $\pi(\mathbf{z})$. Therefore, Eq. (17) guarantees that if $\mathbf{z}$ is from $\pi(\mathbf{z})$, so will be $\mathbf{y}$.

For an aperiodic, irreducible and positive-recurrent Markov chain (i.e. satisfying the above three conditions), the limiting distribution of successive iterates will reach the stationary (target) pdf, regardless of the starting value of the chain[6, 7]. The number of samples required to reach the stationary distribution from a starting sample $\theta_0$ is called the *burn-in period* of the chain. After the burn-in period, the samples of the Markov chain approximately follow the target pdf $\pi$[6, 7].

#### 2.3.2 The Metropolis-Hastings algorithm

Given a transition kernel $A(\mathbf{z},\mathbf{y})$ for a 1st-order aperiodic, irreducible and positive-recurrent Markov chain, its stationary distribution $\pi$ satisfies Eq. (17). In Bayesian inference problems, the target (posterior) distribution is available, but the corresponding transition kernel is not known *a priori*. The method first proposed by Metropolis *et al.* [21] and generalized by Hastings[22] is adopted in this paper.

Starting from an arbitrary sample, a candidate point $\mathbf{y}$ is generated from an arbitrarily chosen proposal pdf $q(\mathbf{z},\mathbf{y})$ in Metropolis-Hastings algorithm. Then the candidate point $\mathbf{y}$ is accepted with probability $\alpha(\mathbf{z},\mathbf{y})$ where

$$\alpha(\mathbf{z},\mathbf{y}) = \min\left(1, \frac{\pi(\mathbf{y})}{\pi(\mathbf{z})}\right). \tag{18}$$

For the Metropolis algorithm[21], the proposal pdf $q(\mathbf{z},\mathbf{y})$ is chosen to be symmetric, i.e. $q(\mathbf{z},\mathbf{y}) = q(\mathbf{y},\mathbf{z})$. This constraint is however relaxed in the Metropolis-Hastings algorithm[22], in which case the acceptance probability is

$$\alpha(\mathbf{z},\mathbf{y}) = \min\left(1, \frac{\pi(\mathbf{y})\,q(\mathbf{y},\mathbf{z})}{\pi(\mathbf{z})\,q(\mathbf{z},\mathbf{y})}\right). \tag{19}$$

Appendix A explains why the above choice of acceptance probability $\alpha(\mathbf{z},\mathbf{y})$ leads to the target pdf $\pi$. The Metropolis-Hastings algorithm has the following attractive features: (1) the normalization constant $p(\mathbf{d})$ in Eq. (1) is not needed; (2) the candidate point $\mathbf{y}$ can be generated from any proposal distribution $q(\mathbf{z},\mathbf{y})$.

## 3 NUMERICAL STUDY - SELF-SUSTAINED OSCILLATIONS OF AN AIRFOIL

An empirical model of the aeroelastic dynamics of a NACA 0012 airfoil is proposed in the form of a generalized Duffing-van der Pol oscillator. We model a wing confined to pure rotation only and forced by random input by the following 3rd order model:

$$\begin{aligned}\ddot{\theta} =& a_1 + a_2\theta + a_3\dot{\theta} + a_4\left|\dot{\theta}\right| + a_5\theta^2 + a_6\theta\dot{\theta} + a_7\dot{\theta}^2 \\ &+ a_8\theta^3 + a_9\theta^2\dot{\theta} + a_{10}\theta\dot{\theta}^2 + a_{11}\dot{\theta}^3 + a_{12}\xi(t)\end{aligned} \tag{20}$$

with $\theta$ is the pitch angle of the wing, $\xi(t)$ is a Gaussian white noise describing the random input (additive modeling error) and $a_{12}$ denotes its strength. We would like to estimate $a_1 \ldots, a_{12}$ using Bayesian inference from a set of noisy observational data obtained at discrete times $t_k$:

$$d_k = \theta(t_k) + \varepsilon_k \tag{21}$$

where $\varepsilon_k$ is white Gaussian measurement noise. We will look at two cases. The first involves assimilating data from one five second long numerical experiment to obtain joint distributions of these parameters. The second case involves assimilating data from four independent experiments, each being five seconds long.

For numerical investigation purposes, the following numerical values of the system parameters are considered: $a_1 = 0$, $a_2 = -540$, $a_3 = 4$, $a_4 = -0.05$, $a_5 = -180$, $a_6 = 0$, $a_7 = 0$, $a_8 = 2.4 \times 10^4$, $a_9 = -2.2 \times 10^3$, $a_{10} = 0$, $a_{11} = 0$ and the intensity of model noise was chosen to be $a_{12} = 0.07$. These parameter values lead to self-sustained oscillations with frequency and amplitude typical of experimental observations[2–4]. The time integration step was chosen to be $\Delta t = 2 \times 10^{-4}$ and the variance of the measurement noise is $5 \times 10^{-6}$. Fig. 1 displays a sample trajectory of the system as well as the corresponding measured pitch from which we would like to estimate the model parameters. The sampling rate for the measurements is 1 kHz as is for the wind-tunnel experiments. Initial conditions of $\theta_0 = 0.0034$ rad and $\dot{\theta}_0 = -0.048$ rad/s were used. The effect of model noise manifests itself in the response as both amplitude and frequency modulation, where the measurement noise induces uncorrelated small amplitude fluctuations.

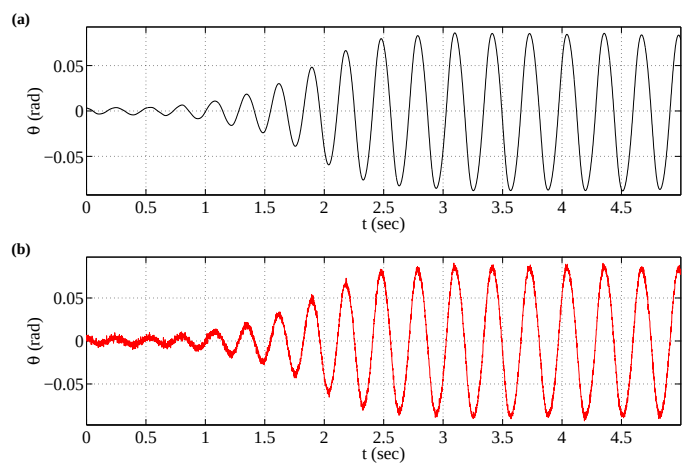


**FIGURE 1**. RESPONSE OF THE PERTURBED NONLINEAR OSCILLATOR: (A) SAMPLE TRAJECTORY AND (B) MEASURED TRAJECTORY

### 3.1 Parameter estimation

The parameters $a_1$ through $a_{12}$ will be estimated using Bayesian inference. The posterior pdf of the unknown parameter is provided in general terms in Eq. (4) and the conditional pdf $p(\mathbf{x}_k = x_k|\theta, \mathbf{d}_{1:k-1} = d_{1:k-1})$ is obtained using the extended Kalman filter. Three cases will be examined. In the first case, data obtained from one five second long numerical experiment will be used to obtain joint distributions of these parameters using MCMC sampling technique. The second case involves assimilating data from four independent experiments, each being five seconds long. The last case will deal with data obtained

 

from wind tunnel experiment.

**3.1.1 case 1** Using the measured response shown in Fig. 1b and a flat prior for the parameters (i.e. $p(\theta) \propto 1$ in Eq. (4)), we obtain the posterior marginal pdfs for the parameters shown in Fig. 2. The pdfs shown are normalized histogram obtained from 8.5 million MCMC samples extracted from 170 MCMC chains running in parallel on a distributed-memory multiprocessor machine (HP Intel Xeon cluster with 178 processor cores) using message-passing interface (MPI) [23]. The dashed line indicates the true parameter values. We can see that the modes of the pdfs does not in general coincide with the true parameter values. This is due to the limited information that the measured data contains regarding these parameters. The marginal pdf with the most non-Gaussian traits is the pdf for the model noise intensity $a_{12}$, being a skewed pdf. The other parameters have close to Gaussian pdfs.

We also obtained the posterior joint pdfs for the parameters. Most of the joint distributions between parameter pairs showed little correlation. For the cases where high correlation was observed (i.e. correlation coefficient greater than 0.4), the joint distributions are shown in Fig. 3. The high correlation between some parameters as shown in Fig. 3 can be explained physically. For example, the high negative correlation between the linear stiffness coefficient $a_3$ and a nonlinear stiffness coefficient $a_8$ is expected as the total stiffness in the system is fixed and thus if one coefficient is overestimated, the other will tend to be underestimated. Similar explanations can be presented to describe the high correlation between some parameters.

**3.1.2 case 2** For this case, we will assimilate data obtained from four independent numerical experiments, each one sampled at a rate of 1 kHz. The overall posterior pdf of the parameters will be proportional to the product of the four posteriors obtained from data for each experiment. The four trajectories and the measured responses are all concatenated and shown in Fig. 4. Once more, a flat prior for the parameters (i.e. $p(a_i) \propto 1$ is utilized to obtain the marginal posterior pdfs shown in Fig. 5. In comparison to the marginal distributions obtained using measurements from just one experiment (case 1) as shown in Fig. 2, the uncertainty in the parameter estimates (measured in terms of support of the pdfs) decreased for all parameters. Furthermore, if the mode of the marginal pdfs is chosen as an estimate for the parameters, the bias in the estimate also decreases when data from four experiments was assimilated in comparison to data from just one experiment. Just as in case 1, the joint distributions highly correlated parameters (i.e. correlation coefficient greater than 0.4) are shown in Fig. 6. It is interesting to note that assimilating more data does not change the shape of the joint distributions (i.e. the correlation coefficients do not vary significantly).

## 4 CONCLUSION

Bayesian inference provides parameter estimates for a self-sustaining aeroelastic oscillator. An empirical model of the aeroelastic system is proposed in the form of a generalized

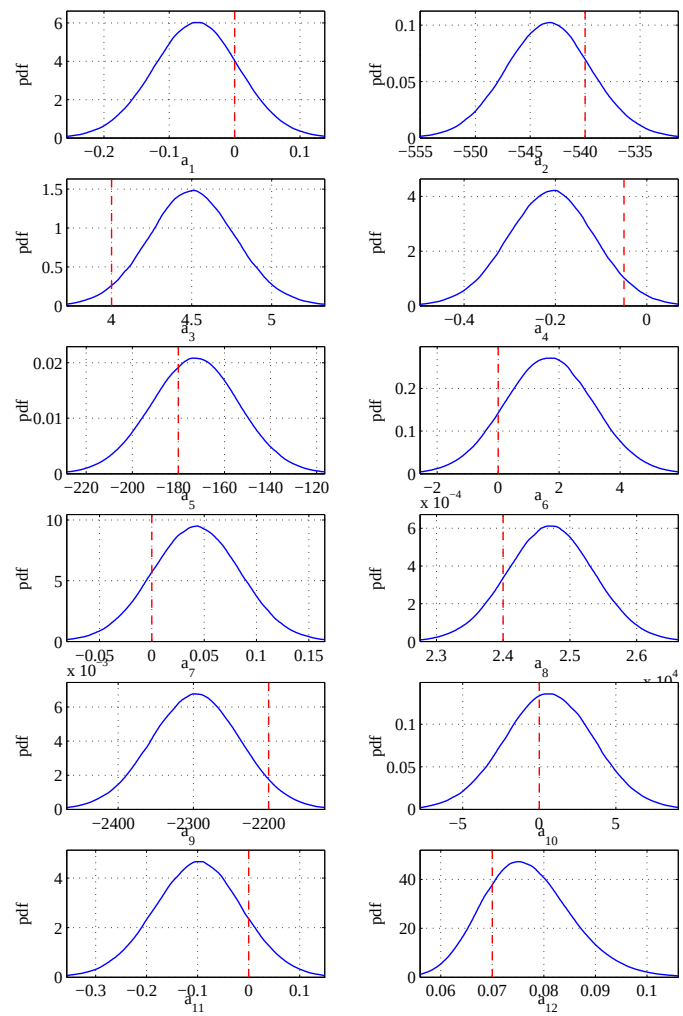


**FIGURE 2**. POSTERIOR PARAMETER PDFS OBTAINED USING MCMC SIMULATIONS FOR CASE 1. DASHED LINE INDICATES TRUE PARAMETER VALUE.

Duffing-van der Pol oscillator, whereby the linear and nonlinear aeroelastic terms were estimated. The model (input) noise process accounting for the amplitude modulation observed from experiments is also estimated. Markov Chain Monte Carlo sampling technique is used to generate samples from the joint distribution of the unknown parameters given noisy measurement data. The extended Kalman filter is utilized to obtain the conditional distribution of the model state given the noisy measurements. The parameters of a numerical third order generalized Duffing-van der Pol oscillator are estimated and marginal and joint probability density functions are presented. In comparison to the case in which data from one experiment was assimilated, assimilating data from four independent experiments reduced both the uncertainty as well as the bias in the estimates.

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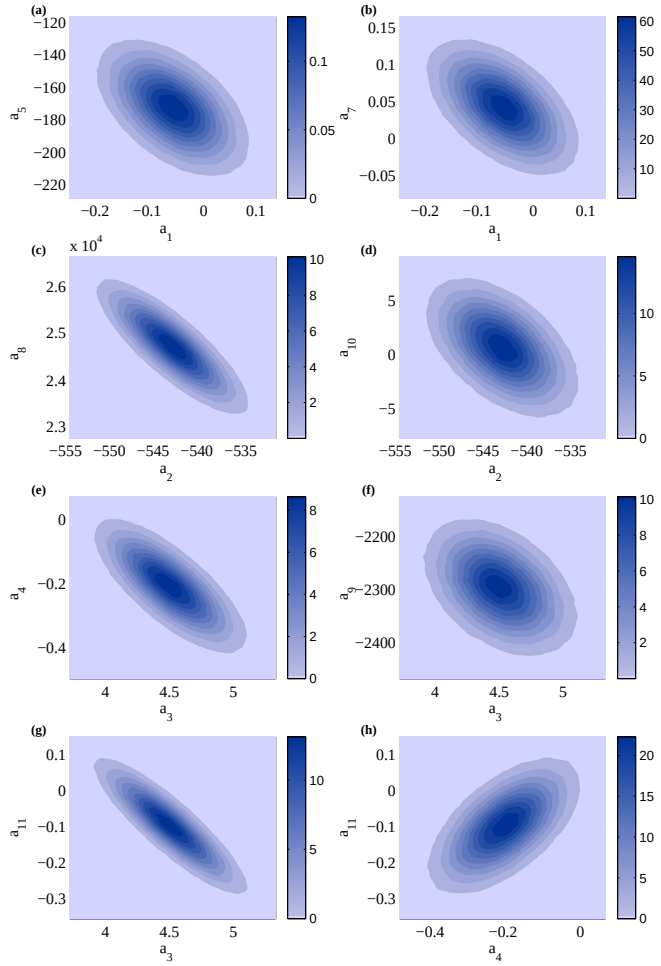


**FIGURE 3**. SELECTED POSTERIOR JOINT PARAMETER PDFS OBTAINED USING MCMC SIMULATIONS FOR CASE 1.

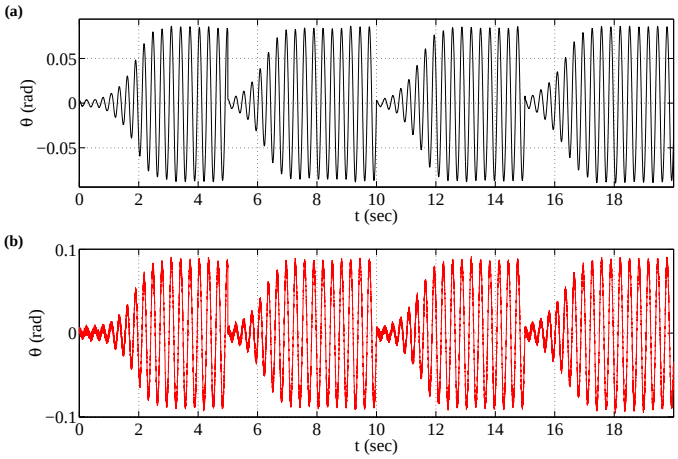


**FIGURE 4**. RESPONSE OF THE PERTURBED NONLINEAR OSCILLATOR FOR 4 INDEPENDENT TRAJECTORIES: (A) SAMPLE TRAJECTORIES AND (B) MEASURED TRAJECTORIES

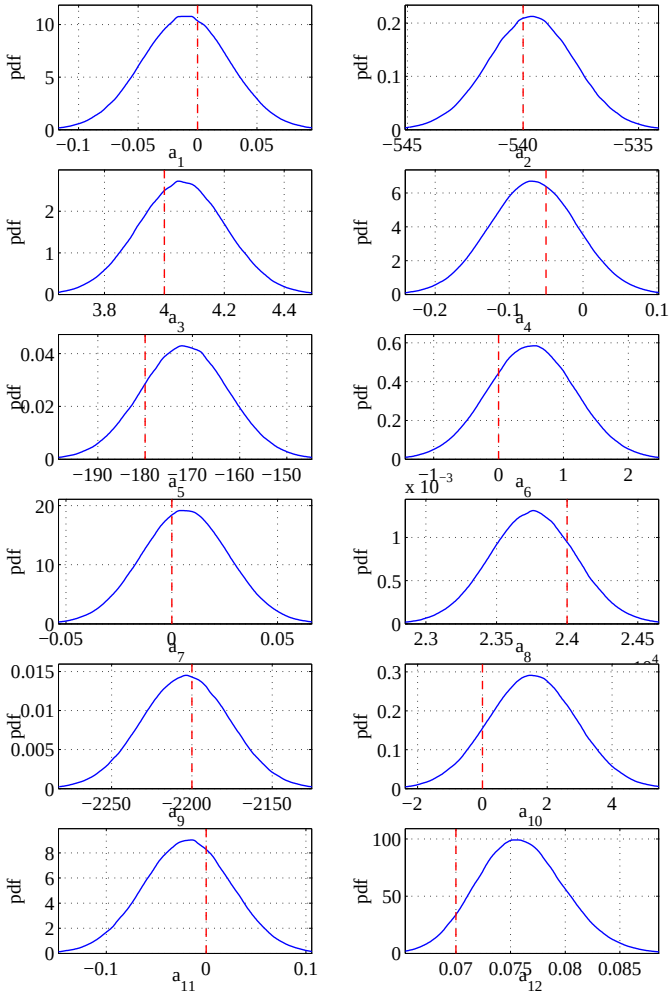


**FIGURE 5**. POSTERIOR PARAMETER PDFS OBTAINED USING MCMC SIMULATIONS FOR CASE 2. DASHED LINE INDICATES TRUE PARAMETER VALUE.

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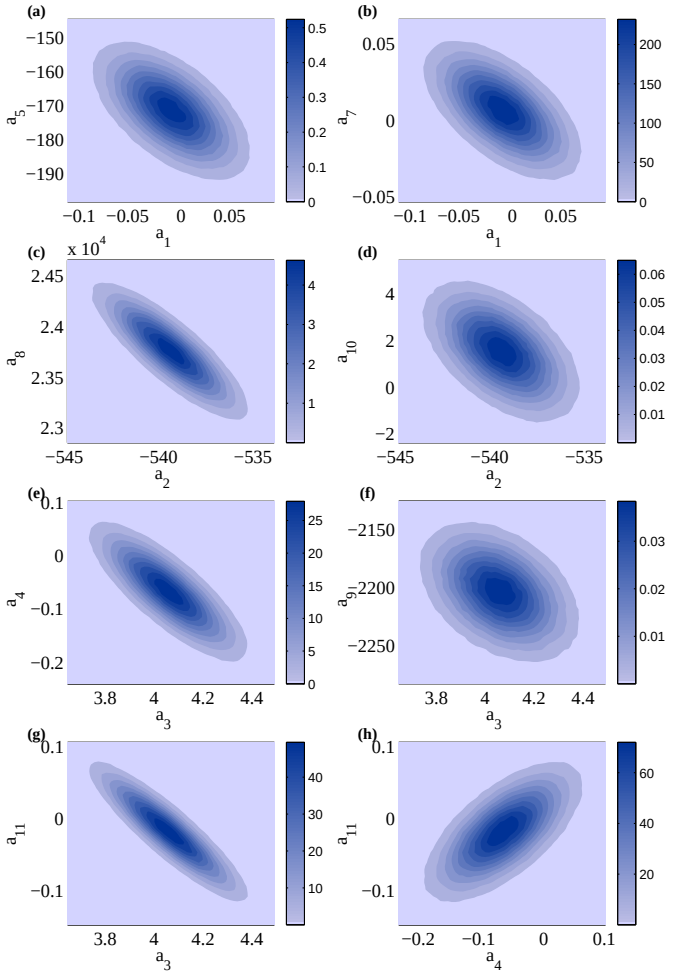


**FIGURE 6**. SELECTED POSTERIOR JOINT PARAMETER PDFS OBTAINED USING MCMC SIMULATIONS FOR CASE 2.


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