Proceedings of the ASME 2010 3rd Joint US-European Fluids Engineering Summer Meeting and 8th International Conference on Nanochannels, Microchannels, and Minichannels
FEDSM-ICNMM2010
August 1-5, 2010, Montreal, Canada

FEDSM-ICNMM2010-30041

APPLICATION OF BAYESIAN INFERENCE TO THE FLUTTER MARGIN METHOD - NEW DEVELOPMENTS

Mohammad Khalil*
Abhijit Sarkar
Department of Civil and Environmental Engineering
Carleton University
Ottawa, Ontario, K1S 5B6
Canada
Email: mkhalil2@connect.carleton.ca

Dominique Poirel
Department of Mechanical Engineering
Royal Military College of Canada
Kingston, Ontario, K7K 7B4
Canada
Email: poirel-d@rmc.ca

ABSTRACT

Zimmerman and Weissenburger flutter margin method is extended to account for modal parameter uncertainties by applying a Bayesian estimation technique to obtain the probability distribution function of the flutter speed. In previous work, a least-squares estimation technique was applied to obtain the posterior pdf of the flutter speed. The limitation of this technique is the assumption that the flutter margin at each airspeed is strictly Gaussian. In this paper, the joint distribution of the modal parameters (and consequently the flutter margin) is obtained from preflutter measured system responses using a full Bayesian analysis utilizing Markov Chain Monte Carlo sampling technique. The flutter margin pdfs are then utilized to obtain the posterior probability density function of the flutter speed. Results are presented for a two-degrees-of-freedom numerical model, for which the true flutter speed is known.

INTRODUCTION

The flutter margin, a quantity first utilized by Zimmerman and Weissenburger [1], is a measure of stability of an aeroelastic system. On the basis that the instability mechanism is dictated by the coalescence of the two modes of vibration that participate in flutter, the typical section model (two degrees-of-freedom (DOF), in pitch and heave) is used. The flutter margin is equivalent to the third sub-determinant of the fourth-order characteristic equation of the typical section model. Hence, according to the Routh's stability criteria, a necessary condition for stability is that the flutter margin must be positive. The flutter margin is given in terms of ω_1, ω_2 and β_1, β_2 being the modal frequencies and negative of decay rates, respectively, of the two modes that participate in flutter:

$$\begin{aligned} F = &\left[\left(\frac{\omega_2^2-\omega_1^2}{2}\right)+\left(\frac{\beta_2^2-\beta_1^2}{2}\right)\right]^2 \\ &+\ 4\beta_1\beta_2\left[\left(\frac{\omega_2^2+\omega_1^2}{2}\right)+2\left(\frac{\beta_2+\beta_1}{2}\right)^2\right] \\ &-\ \left[\left(\frac{\beta_2-\beta_1}{\beta_2+\beta_1}\right)\left(\frac{\omega_2^2-\omega_1^2}{2}\right)+2\left(\frac{\beta_2+\beta_1}{2}\right)^2\right]^2. \end{aligned} \tag{1}$$

The flutter margin can be used to extrapolate the classical flutter speed from pre-flutter flight test data. Under modest restrictions, the flutter margin varies parabolically in dynamic pressure if structural damping is neglected. The inclusion of structural damping in the analysis leads to a sixth-order relationship between the flutter margin and airspeed. For larger airspeeds the relationship converges to a fourth-order polynomial, as per Eq. (2), since aerodynamic damping dominates structural damping.

$$F = b_1U^4 + b_2U^2 + b_3. \tag{2}$$

The coefficients, b_1 through b_3, can be estimated using a least-square fitting procedure of Eq. (2) with at least three subcritical airspeeds. The flutter speed, U_f, is the root of Eq. (2).

*Address all correspondence to this author.

Since its establishment in 1964, the flutter margin method has been generalized to tackle various cases overlooked by the basic method. Price and Lee [2] extended the method to trinary flutter, for instance. Recently, the uncertainty in the modal parameters has been propagated by Poirel *et al.* [3] to provide a probability density function (pdf) of the flutter margin and consequently the flutter speed. The basic idea is to directly carry through the analysis the modal parameter uncertainties to construct flutter margin and flutter speed histograms from which different statistics are calculated. It was shown that although the modal parameters were assumed to be Gaussian, the flutter speed pdf was not symmetric, but skewed. This methodology was further examined by Heeg [4]. A maximum likelihood parameter estimation method was used to construct a pdf of the flutter speed. Khalil *et al.* [5] provided a pdf of the flutter speed by estimating the joint distribution of the flutter margin equation coefficients in Eq. (2). The flutter margin pdfs were obtained from modal parameter estimates which were artificially perturbed by Gaussian noise.

In this paper, the method developed by Khalil *et al.* [5] is further developed as a number of refinements are possible. Here, the joint distribution of the modal parameters is obtained from measured noisy free decay responses of the system. Bayesian inference provides a pdf of these parameters and Markov Chain Monte Carlo (MCMC) sampling technique [6, 7] is utilized to provide samples from this pdf. The flutter margin pdf is then constructed from these modal parameter samples. Finally, the joint distribution of the flutter margin equation coefficients in Eq. (2) (and consequently the flutter speed) is obtained again using Bayesian inference. For numerical illustration, the proposed method is applied for a two-degrees-of-freedom numerical model, for which the true flutter speed is known.

BAYESIAN INFERENCE

Bayesian inference provides a statistical framework which relates physical observations to mathematical models[6–8]. Let $\mathbf{x}$ be the vector of system parameters and the vector $\mathbf{d}$ denotes a collection of physical observations. In Bayesian settings, the conditional pdf $p(\mathbf{d}|\mathbf{x})$ of the observations given the system parameters is called the *forward* pdf or likelihood function. When observations are available, Bayesian inference deduces the value of the system parameters relying on a valid mathematical model. This step provides the conditional pdf $p(\mathbf{x}|\mathbf{d})$ of the system parameters given the observations, denoted by the *inverse* or posterior pdf. Bayes theorem relates the inverse and forward pdf[6–8]:

$$p(\mathbf{x}|\mathbf{d}) = \frac{p(\mathbf{d}|\mathbf{x})\,p(\mathbf{x})}{p(\mathbf{d})}. \tag{3}$$

In Eq. (3), $p(\mathbf{x})$ is the pdf that contains the available knowledge of the parameters before making the observation $\mathbf{d}$[6–8]. It is called the *prior* pdf of $\mathbf{x}$. On the left-hand side of Eq. (3), we have the *posterior* pdf $p(\mathbf{x}|\mathbf{d})$ of $\mathbf{x}$ that represents our knowledge of the system parameters after making the observation. To obtain the posterior pdf, the prior pdf is multiplied by the forward or *likelihood* pdf $p(\mathbf{d}|\mathbf{x})$ which can be characterized by the mathematical model of the system. The factor $p(\mathbf{d}) = \int p(\mathbf{d}|\mathbf{x})\,p(\mathbf{x})\,\mathrm{d}\mathbf{x}$ is just a normalization constant.

In many cases of practical interest, the analytical expression for the posterior pdf is not generally available, since (1) the prior probability function $p(\mathbf{x})$ may involve information which is difficult to express in analytical form and (2) the normalization constant $p(\mathbf{d})$ is hard to evaluate. Even when the analytical expression for the posterior pdf is available, the integration of the posterior pdf becomes difficult to evaluate when the number of components in $\mathbf{x}$ is large. One way to alleviate this difficulty is to use random sampling techniques such as MCMC which is adopted in this paper.

MARKOV CHAIN MONTE CARLO SAMPLING

Simple Monte Carlo sampling technique (e.g. [6]) can generate *independent* random samples from the posterior pdf $p(\mathbf{x}|\mathbf{d})$ from which one can extract relevant statistical information (e.g. mean and mode). For a general non-Gaussian posterior pdf, independent samples are difficult to simulate in practice. One can however sample from a Markov chain whose *equilibrium* or *stationary* pdf matches the posterior pdf $p(\mathbf{x}|\mathbf{d})$ as described in the following Subsection. Although the samples are not independent, the statistical features of the posterior pdf can be estimated with reasonable accuracy using a large enough sample size. In this Section, we briefly describe Markov chains and present the general algorithm of MCMC adopted in this paper, namely the Metropolis algorithm. More details can be obtained in the books by Gilks *et al.* [6] and Liu [7].

Markov Chains

A sequence of random vectors $\{\mathbf{x}_0, \mathbf{x}_1, \ldots\}$ is called a first-order Markov chain if it satisfies the property[7]

$$p(\mathbf{x}_{i+1} = \mathbf{y}|\mathbf{x}_i = \mathbf{x}, \ldots, \mathbf{x}_0 = \mathbf{z}) = p(\mathbf{x}_{i+1} = \mathbf{y}|\mathbf{x}_i = \mathbf{x}). \tag{4}$$

It states that $\mathbf{x}_{i+1}$ statistically depends only on the previous random vector $\mathbf{x}_i$. When the transition pdf (kernel) $p(\mathbf{x}_{i+1} = \mathbf{y}|\mathbf{x}_i = \mathbf{x})$ is time-invariant (independent of i), it can be denoted by $A(\mathbf{x}, \mathbf{y})$.

The pdf $p(\mathbf{x}_i)$ converges to a stationary pdf $\pi(\mathbf{x}_i)$ as $i \to \infty$ if the chain satisfies three properties[6]: (1) the chain is irreducible, i.e. starting with any possible state for $\mathbf{x}_0$ the chain can reach any other state in some number of iterations; (2) the chain is aperiodic, i.e. the chain does not periodically oscillate among different sets of states; (3) the chain is positive recurrent, i.e. if the value of $\mathbf{x}_0$ is sampled from the stationary distribution π, all subsequent iterates will be distributed according to π. The last condition is met if the stationary pdf π and the transition kernel $A(\mathbf{x}, \mathbf{y})$ satisfy the following eigenvalue problem:

$$\int \pi(\mathbf{x})\,A(\mathbf{x}, \mathbf{y})\,\mathrm{d}\mathbf{x} = \pi(\mathbf{y}). \tag{5}$$

The left-hand side of Eq. (5) gives the marginal distribution of $\mathbf{y}$ under the assumption that $\mathbf{x}$ is from $\pi(\mathbf{x})$. Therefore, Eq. (5) guarantees that if $\mathbf{x}$ is from $\pi(\mathbf{x})$, so will be $\mathbf{y}$.

For an aperiodic, irreducible and positive-recurrent Markov chain (i.e. satisfying the above three conditions), the limiting distribution of successive iterates will reach the stationary (target) pdf, regardless of the starting value of the chain[6, 7]. The number of samples required to reach the stationary distribution from a starting sample $\mathbf{x}_0$ is called the *burn-in period* of the chain. After the burn-in period, the samples of the Markov chain approximately follow the target pdf π[6, 7].

The Metropolis-Hastings algorithm

Given a transition kernel $A(\mathbf{x},\mathbf{y})$ for a 1st-order aperiodic, irreducible and positive-recurrent Markov chain, its stationary distribution π satisfies Eq. (5). In Bayesian inference problems, the target (posterior) distribution is available, but the corresponding transition kernel is not known *a priori*. The method first proposed by Metropolis *et al.* [9] and generalized by Hastings[10] is adopted in this paper.

Starting from an arbitrary sample, a candidate point $\mathbf{y}$ is generated from an arbitrarily chosen proposal pdf $q(\mathbf{x},\mathbf{y})$ in Metropolis-Hastings algorithm. Then the candidate point $\mathbf{y}$ is accepted with probability $\alpha(\mathbf{x},\mathbf{y})$ where

$$\alpha(\mathbf{x},\mathbf{y}) = \min\left(1, \frac{\pi(\mathbf{y})}{\pi(\mathbf{x})}\right). \tag{6}$$

For the Metropolis algorithm[9], the proposal pdf $q(\mathbf{x},\mathbf{y})$ is chosen to be symmetric, i.e. $q(\mathbf{x},\mathbf{y}) = q(\mathbf{y},\mathbf{x})$. This constraint is however relaxed in the Metropolis-Hastings algorithm[10], in which case the acceptance probability is

$$\alpha(\mathbf{x},\mathbf{y}) = \min\left(1, \frac{\pi(\mathbf{y})\,q(\mathbf{y},\mathbf{x})}{\pi(\mathbf{x})\,q(\mathbf{x},\mathbf{y})}\right). \tag{7}$$

Appendix A explains why the above choice of acceptance probability $\alpha(\mathbf{x},\mathbf{y})$ leads to the target pdf π. The Metropolis-Hastings algorithm has the following attractive features: (1) the normalization constant $p(\mathbf{d})$ in Eq. (3) is not needed; (2) the candidate point $\mathbf{y}$ can be generated from any proposal distribution $q(\mathbf{x},\mathbf{y})$.

APPLICATION OF BAYESIAN INFERENCE TO FLUTTER SPEED ESTIMATION

This Section describes the application of Bayesian inference for flutter speed estimation problem. For given airspeeds U_i, $i = 1,\ldots,n_u$, one can estimate the joint distribution of the modal parameters of the two modes involved in coalescence flutter from noisy measurements of free decay system response using Bayesian inference. Assuming the typical section model (two degrees-of-freedom (DOF), in pitch θ and heave h) is utilized, the problem involves identifying the parameters $a_1,\ldots,a_{12}$ that describe the free decay response of the system given unknown initial conditions (please refer to [11] for the derivation of this representation):

$$h(t) = a_1 e^{-a_{10}t}\cos(a_9 t + a_5) + a_2 e^{-a_{12}t}\cos(a_1 1 t + a_6) \tag{8}$$

$$\theta(t) = a_3 e^{-a_{10}t}\cos(a_9 t + a_7) + a_4 e^{-a_{12}t}\cos(a_1 1 t + a_8) \tag{9}$$

from noisy measurements

$$h_k = a_1 e^{-a_{10}t_k}\cos(a_9 t_k + a_5) + a_2 e^{-a_{12}t_k}\cos(a_1 1 t_k + a_6) + n_{1,k} \tag{10}$$

$$\theta_k = a_3 e^{-a_{10}t_k}\cos(a_9 t_k + a_7) + a_4 e^{-a_{12}t_k}\cos(a_1 1 t_k + a_8) + n_{2,k} \tag{11}$$

where $n_{1,k}$ and $n_{2,k}$ are independent and identically distributed Gaussian noise with zero mean and variance equal to γ_1 and γ_2, respectively.

The likelihood function of these unknown parameters is

$$\begin{aligned} &p(a_1,\ldots,a_{12}|h_1,\ldots,h_N,\theta_1,\ldots,\theta_N) \\ &= \frac{1}{(2\pi\gamma_1)^{N/2}}\exp\left[\sum_{k=1}^{N} -\frac{1}{2\gamma_1}\left(h_k - \hat{h}_k(a_1,\ldots,a_{12})\right)^2\right] \\ &\times \frac{1}{(2\pi\gamma_2)^{N/2}}\exp\left[\sum_{k=1}^{N} -\frac{1}{2\gamma_2}\left(\theta_k - \hat{\theta}_k(a_1,\ldots,a_{12})\right)^2\right] \end{aligned} \tag{12}$$

$\hat{h}_k(a_1,\ldots,a_{12})$ and $\hat{\theta}_k(a_1,\ldots,a_{12})$ are the predicted free decay system responses for the assumed values of $a_1\ldots,a_{12}$. Samples of these parameters (and thus samples of modal parameters $\omega_1 = a_9$, $\omega_2 = a_{11}$ and $\beta_1 = -a_{10}$, $\beta_2 = -a_{12}$) are obtained from this joint pdf using MCMC sampling technique. The modal parameter samples are consequently transformed using Eq. (1) to obtain flutter margin samples from which a flutter margin pdf is constructed. The flutter margin pdf obtained at each experimental airspeed using the proposed method will be denoted by $p(F_i|\mathbf{b})$, being conditional on the unknown coefficients $\mathbf{b} = \{b_1, b_2, b_3\}$ from Eq. (2).

We are interested in estimating the joint distribution of the vector $\mathbf{b} = \{b_1, b_2, b_3\}$ of unknown coefficients. This would in turn provide a pdf for the flutter speed being the root of Eq. (2). Let $\mathbf{F} = \{F_1, F_2, \ldots, F_{n_u}\}$ represent the random vector of the flutter margin at airspeeds U_i, $i = 1,\ldots,n_u$. By Bayes' theorem, we have

$$p(\mathbf{a}|\mathbf{F}) \propto p(\mathbf{F}|\mathbf{b})\,p(\mathbf{b}). \tag{13}$$

We assume that the likelihood function $p(\mathbf{F}|\mathbf{b})$ can be expressed by the product of the marginal pdfs of F_i as

$$p(\mathbf{F}|\mathbf{b}) = \prod_{i=1}^{n_u} p(F_i|\mathbf{b}) \tag{14}$$

and the prior probability function of **a** is expressed as

$$p(\mathbf{b}) \propto \begin{cases} 1 \text{ , if } b_1, b_2, b_3 \text{ satisfy conditions } (1)-(2) \\ 0 \text{ , otherwise} \end{cases} . \quad (15)$$

where

$$\begin{array}{l} \text{condition 1 : } b_2^2 - 4b_1b_3 > 0 \\ \text{condition 2 : } b_3 > 0 \end{array} .$$

The above form for $p(\mathbf{F}|\mathbf{b})$ is valid under the simplifying assumption that the flutter margin at one airspeed is independent from the flutter margin at other airspeeds. As for the prior pdf $p(\mathbf{b})$ in Eq. (15) of the unknown coefficients, it reflects the assumption that the polynomial in Eq. (2) (a) has real roots (condition 1) and (b) provides a positive flutter margin at zero airspeed (condition 2). The two conditions guarantee physically meaningful realizations of the flutter margin polynomial.

As a result, the conditional pdf for **a** in Eq. (13) has a complicated mathematical expression and the normalization constant is hard to obtain. Consequently, MCMC sampling technique is applied to obtain samples from this complex conditional distribution $p(\mathbf{a}|\mathbf{F})$. These would in turn provide samples of the flutter speed U_f, being the root of the flutter margin equation. The specific MCMC sampling technique utilized is the Metropolis-Hastings algorithm which was overviewed previously.

Application to a Two-DOF Airfoil

We apply the proposed methodology to a two-DOF airfoil model with quasi-steady aerodynamics model and compare these results with the conventional flutter margin method. The results are then compared with the known (deterministic) flutter speed, which is 23.69 m/s for the airfoil parameters chosen as determined by eigenvalue analysis. Note that proportional structural damping is assumed with damping ratios equal to 10% for both structural modes of vibration. Five equally spaced airspeeds were chosen for numerical experiments with the lowest airspeed being 65% of the flutter speed and the highest airspeed is at 87% of the flutter speed. The chosen airspeeds for the analysis and the respective modal parameter and flutter margin values are summarized in Tab. 1.

At each airspeed, noisy measurements of the free decay response of the system are obtained over an interval of 0.8 seconds. The initial condition for the system is -0.05 m in heave and 0.175 rad in pitch with velocities equal to zero for both degrees of freedom. The variance of measurement noise for the heave and pitch measurements were chosen to be $\gamma_1 = 2 \times 10^{-5}$ and $\gamma_2 = 2 \times 10^{-4}$, respectively. The measurements were obtained with a sampling rate of 5 kHz. Figs. 1-5 display the true trajectories of the system as well as the corresponding measured responses at all five airspeeds in Tab. 1.

We first apply MCMC sampling technique to obtain jointly distributed samples for the 12 unknown parameters $a_1, \ldots, a_12$ in Eq. (12). At each airspeed, 5×10^7 MCMC samples

TABLE 1. TRUE MODAL PARAMETERS AND FLUTTER-MARGIN FOR 2-DOF AIRFOIL

U (m/s)	ω_1 (rad/s)	β_1 (1/s)	ω_2 (rad/s)	β_2 (1/s)	F ($\times 10^6$)
15.5	55.68	-6.207	12.56	-3.966	2.223
16.75	53.70	-5.934	12.85	-4.431	1.972
18	51.48	-5.553	13.21	-5.004	1.688
19.25	48.99	-5.016	13.62	-5.734	1.372
20.5	46.19	-4.237	14.09	-6.705	1.024

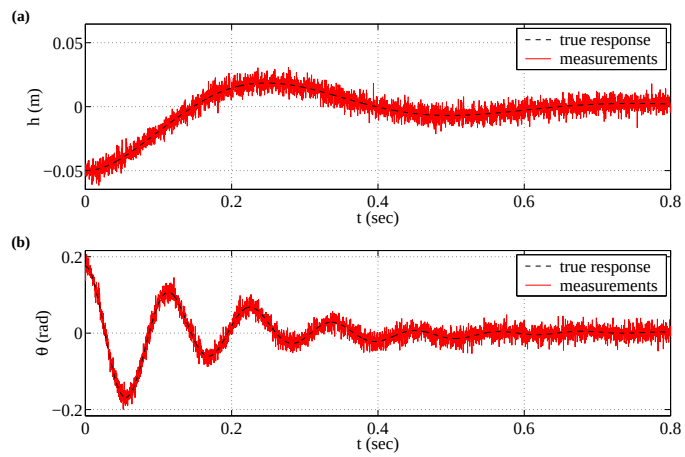

FIGURE 1. (A) HEAVE RESPONSE AND (B) PITCH RESPONSE OF THE 2-DOF AIRFOIL FOR $U = 15.5$ m/s

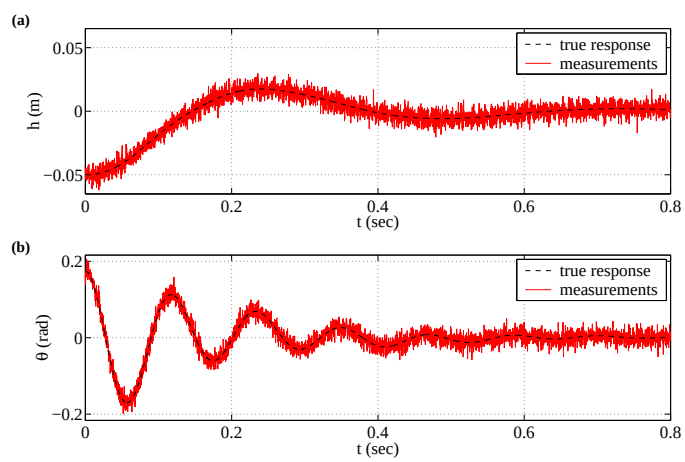

FIGURE 2. (A) HEAVE RESPONSE AND (B) PITCH RESPONSE OF THE 2-DOF AIRFOIL FOR $U = 16.75$ m/s

were obtained with 170 MCMC chains running in parallel on a distributed-memory multiprocessor machine (HP Intel Xeon

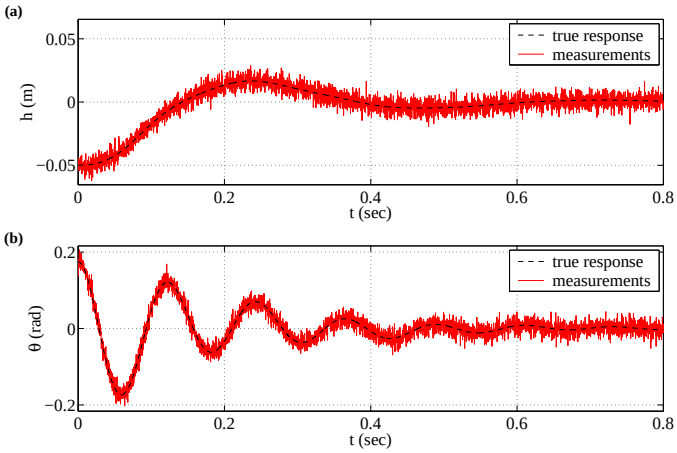

FIGURE 3. (A) HEAVE RESPONSE AND (B) PITCH RESPONSE OF THE 2-DOF AIRFOIL FOR $U = 18$ m/s

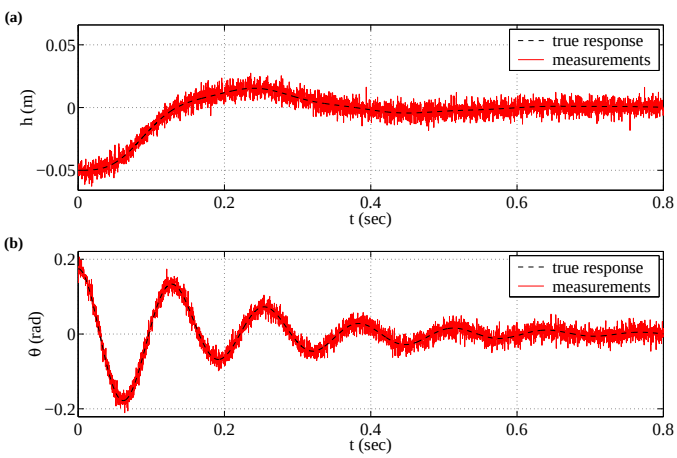

FIGURE 4. (A) HEAVE RESPONSE AND (B) PITCH RESPONSE OF THE 2-DOF AIRFOIL FOR $U = 19.25$ m/s

cluster with 178 processor cores) using message-passing interface (MPI) [12]. From these we can extract the marginal pdfs of each modal parameter at each air speed shown in Figs. 6-10. The dashed line indicates the true modal parameter value. We can see that the mode of the pdfs does not in general coincide with the true parameter value. This is due to the limited information that the measured data contains regarding these parameters. One can apply the same technique to data obtained from more than one experiment at each air speed to decrease this observed bias as well as decreasing the uncertainty in these estimates. One can apply the classical method in which flutter margin values at each airspeed are obtained using the mean modal parameter values. In this case we obtain five flutter margin values at five different airspeeds from which a quadratic polynomial in the square of the airspeed can be obtained using least-squares fitting. The five data points and least-squares fit are shown in Fig. 11. The extracted flutter speed using this method is 23.63 m/s, being close to the

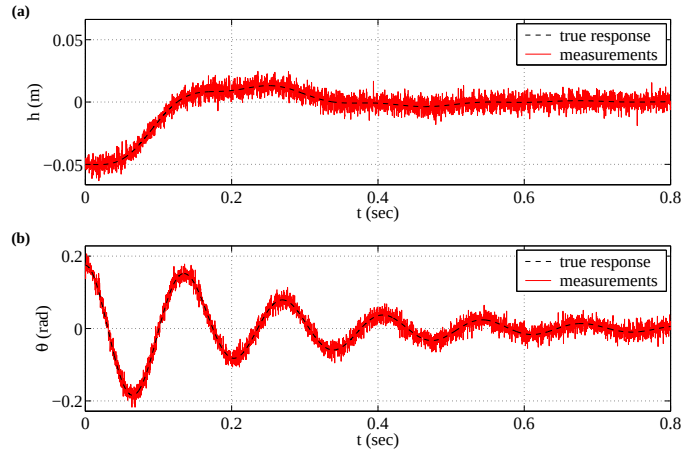

FIGURE 5. (A) HEAVE RESPONSE AND (B) PITCH RESPONSE OF THE 2-DOF AIRFOIL FOR $U = 20.5$ m/s

true flutter speed of 23.69 m/s. The results using the classical method are very accurate and the polynomial fit is good (as seen in Fig. 11). We suspect this would not be the case when a more realistic case is examined in which external random excitations perturb the system.

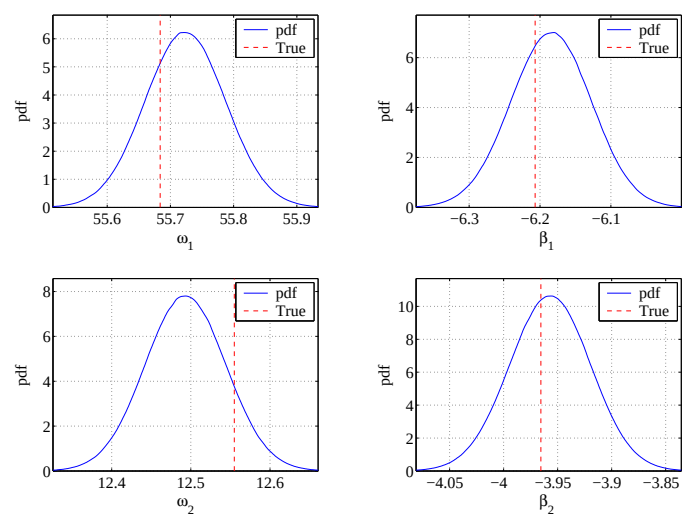

FIGURE 6. MARGINAL PDFS OF MODAL PARAMETERS FOR $U = 15.5$ m/s

One can also examine the joint distributions between the modal parameter estimates. These joint pdfs are presented in Figs. 12-16. One observes a jointly Gaussian nature and small if any correlation between these parameters. Using the MCMC samples obtained from the joint distribution between the four modal parameters, we obtain samples of the flutter margin and construct the flutter margin pdf at each airspeed. These pdfs are shown in Fig. 17. These flutter margin estimates at the five air-

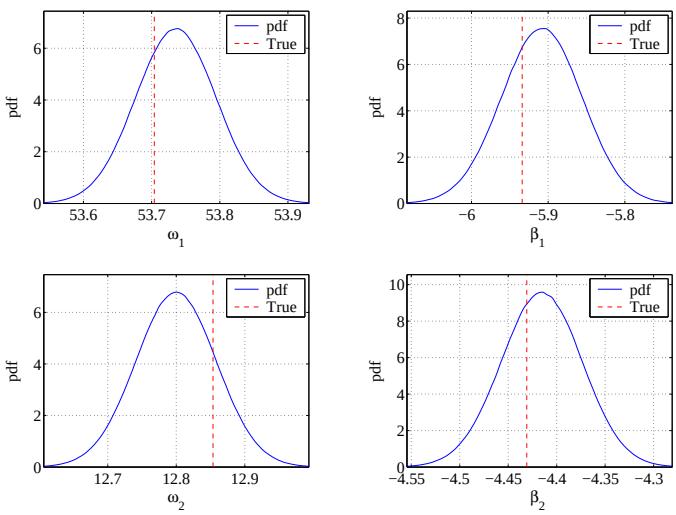

FIGURE 7. MARGINAL PDFS OF MODAL PARAMETERS FOR $U = 16.75$ m/s

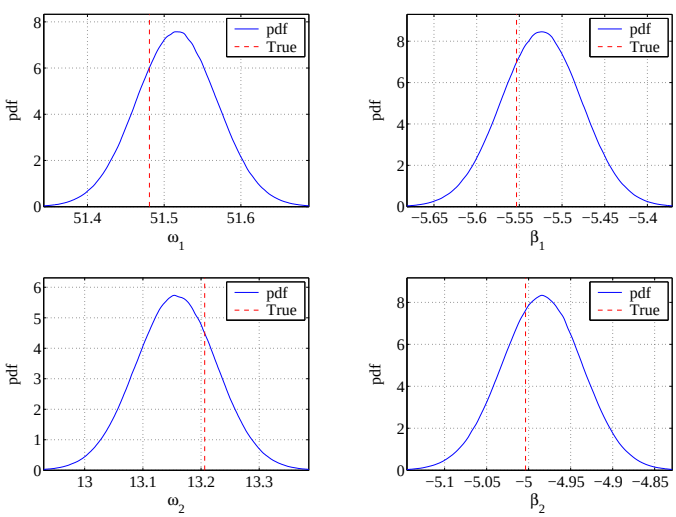

FIGURE 8. MARGINAL PDFS OF MODAL PARAMETERS FOR $U = 18$ m/s

speeds represent $p(F_i|a)$ in Eq. (14). We use MCMC technique to obtain 1.7×10^9 samples of $\{a_1, a_2, a_3\}$ from the posterior pdf in Eq. (13). Each of these samples represent a possible flutter margin polynomial. In turn, 1.7×10^9 samples of the flutter speed are obtained, being the root of the flutter margin polynomial. The histogram of the root of these realizations is shown in Fig. 18. We can see that both the true flutter speed as well as the estimate obtained using the conventional method fall within the support of the flutter speed pdf obtained using Bayesian inference. Furthermore, the range of possible flutter speed values is relatively small considering that the maximum experimental airspeed was 20.5 m/s.

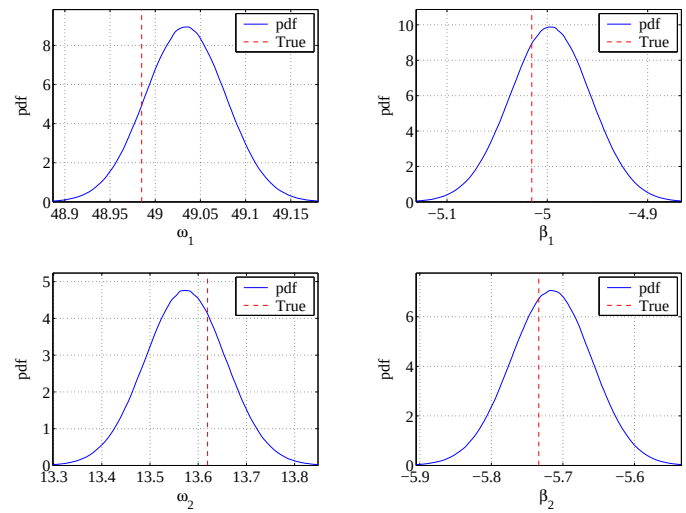

FIGURE 9. MARGINAL PDFS OF MODAL PARAMETERS FOR $U = 19.25$ m/s

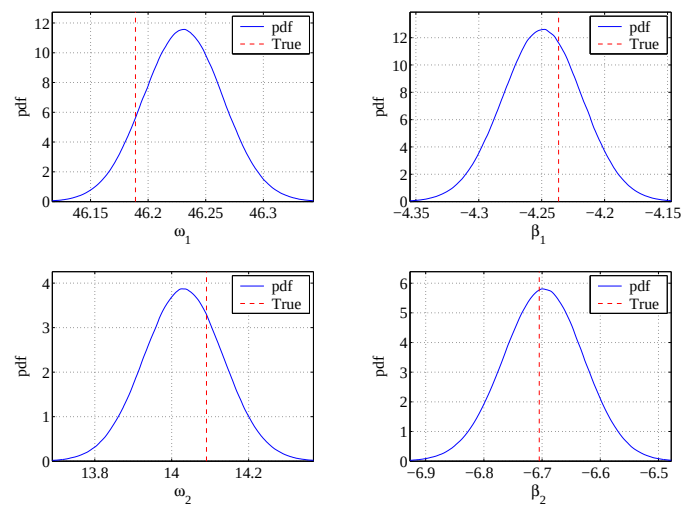

FIGURE 10. MARGINAL PDFS OF MODAL PARAMETERS FOR $U = 20.5$ m/s

CONCLUSION

A Bayesian estimation technique based on the Metropolis-Hastings MCMC algorithm is proposed to obtain the probability distribution function of the flutter speed from modal parameter estimates using the flutter margin method proposed by Zimmerman and Weissenburger. The proposed method is applied to a two-degrees-of-freedom numerical model. For the numerical experiments conducted in this paper, the proposed method provides a reasonable confidence interval for the flutter speed that the conventional method does not provide. As a next step, a more realistic numerical model will be examined in which mode noise is present. Furthermore, the Bayesian methodology used herein will be validated blind using wind tunnel flutter test data in order to mimic a realistic flight test environment. The method will also

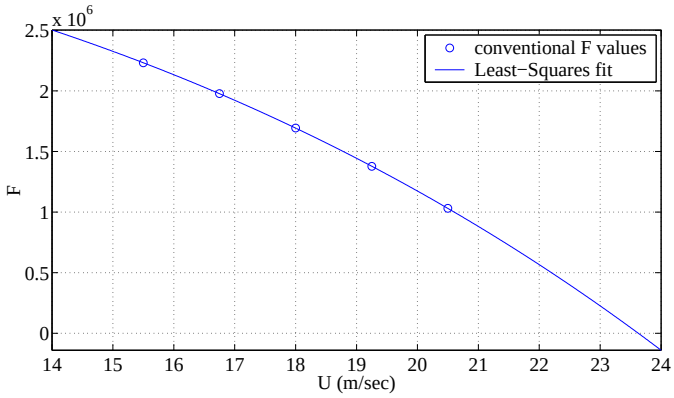

FIGURE 11. CONVENTIONALLY OBTAINED FLUTTER MARGIN VALUES (USING MEAN MODAL PARAMETER ESTIMATES)

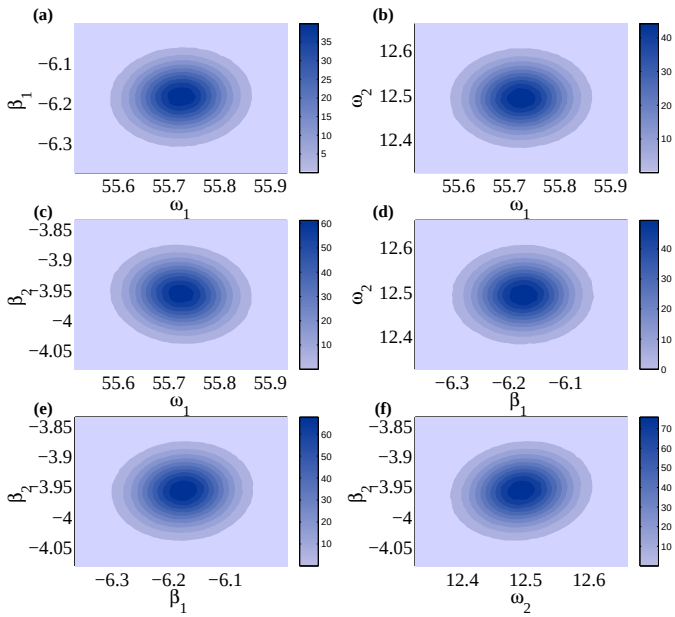

FIGURE 12. JOINT PDFS OF MODAL PARAMETERS FOR $U = 15.5\ \mathrm{m/s}$

be applied to measured response of continuous system including available real F-18 flight test experiments.

ACKNOWLEDGMENTS

The first author acknowledges the support of the Department of National Defense of Canada. The second author acknowledges the support of the Canadian Department of National Defense and the Natural Sciences and Engineering Research Council of Canada. The third author acknowledges the support of a Discovery Grant from Natural Sciences and Engineering Research Council of Canada and the Canada Research Chair Program.

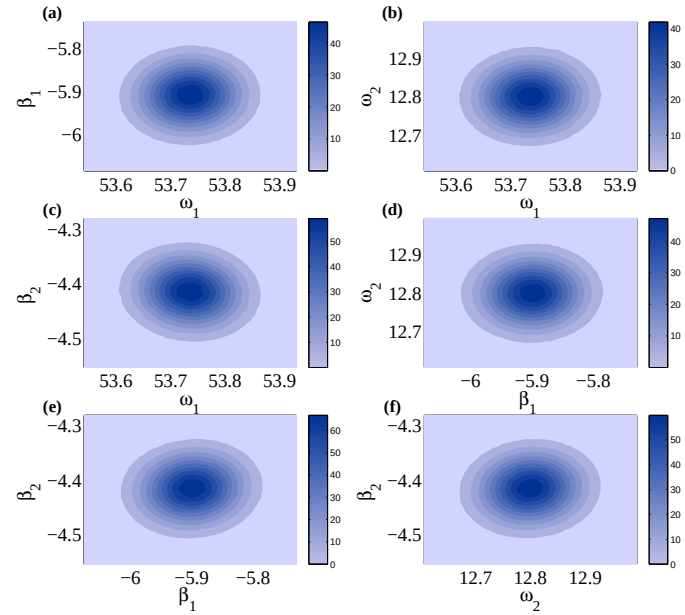

FIGURE 13. JOINT PDFS OF MODAL PARAMETERS FOR $U = 16.75\ \mathrm{m/s}$

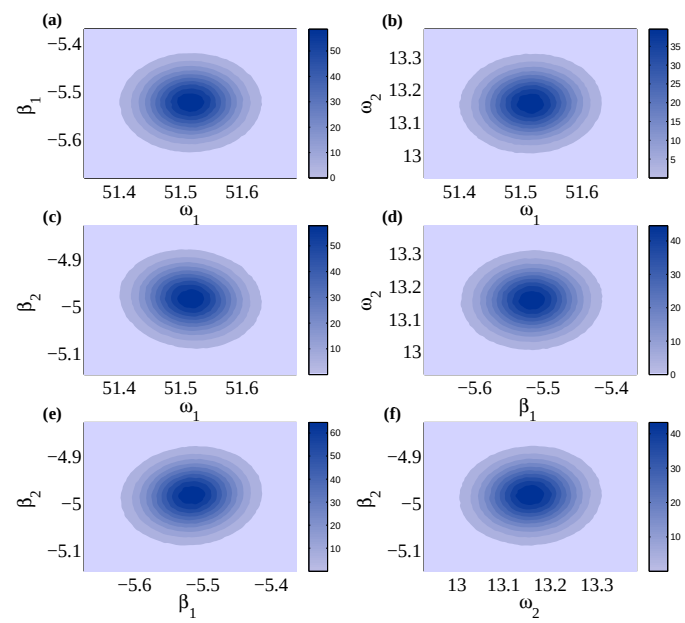

FIGURE 14. JOINT PDFS OF MODAL PARAMETERS FOR $U = 18\ \mathrm{m/s}$

Appendix A

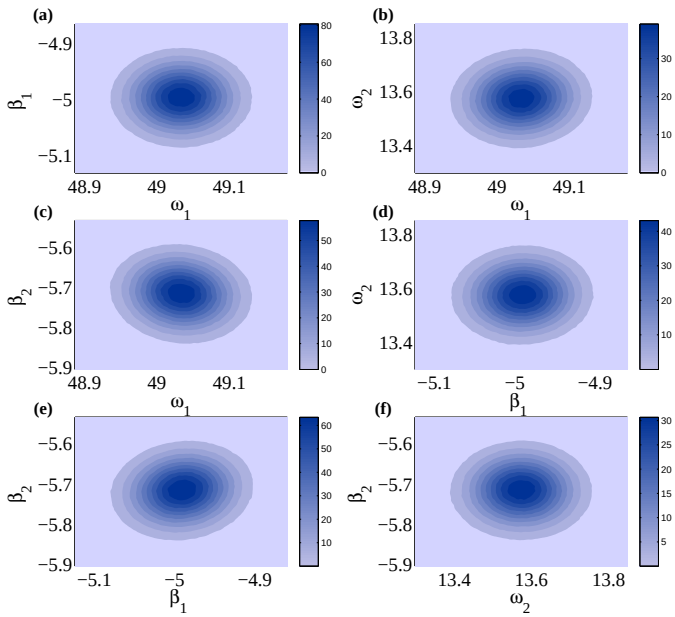

FIGURE 15. JOINT PDFS OF MODAL PARAMETERS FOR $U = 19.25$ m/s

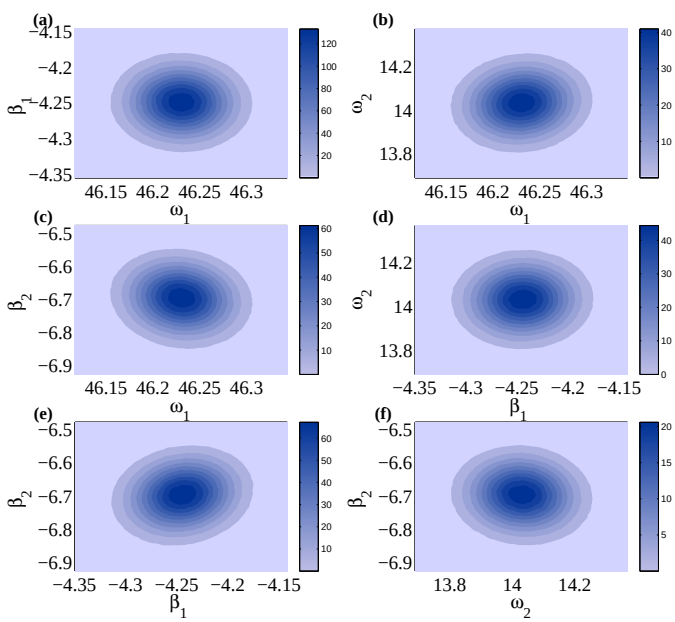

FIGURE 16. JOINT PDFS OF MODAL PARAMETERS FOR $U = 20.5$ m/s

STATIONARY DISTRIBUTION SIMULATED FROM METROPOLIS-HASTINGS ALGORITHM

For the Metropolis-Hastings algorithm[10], the acceptance probability $\alpha(\mathbf{x},\mathbf{y})$ is

$$\alpha(\mathbf{x},\mathbf{y}) = \min\left(1, \frac{\pi(\mathbf{y})\,q(\mathbf{y},\mathbf{x})}{\pi(\mathbf{x})\,q(\mathbf{x},\mathbf{y})}\right) \tag{16}$$

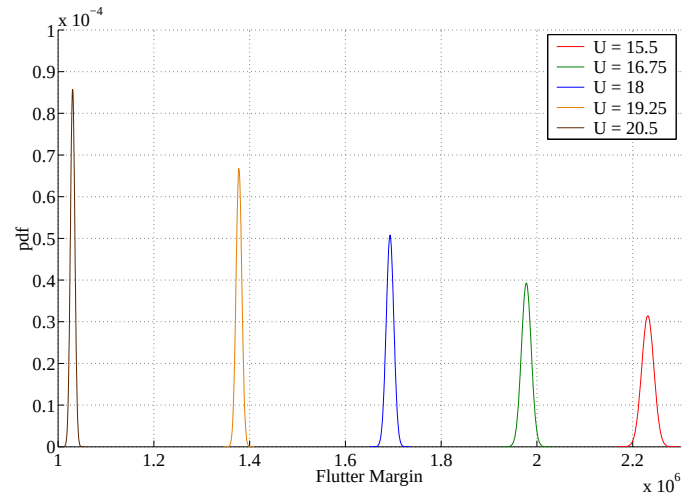

FIGURE 17. FLUTTER MARGIN PDFS

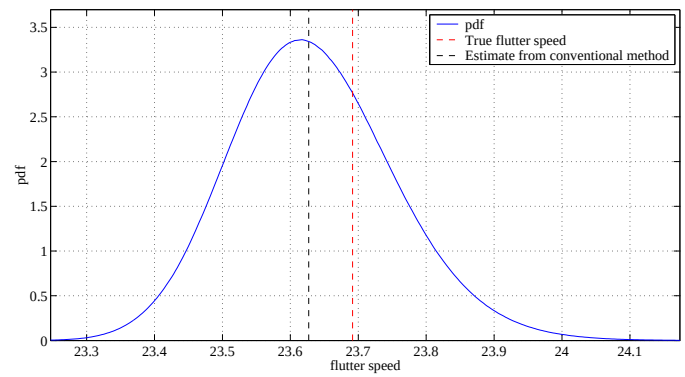

FIGURE 18. FLUTTER SPEED PDF OBTAINED USING BAYESIAN INFERENCE

For this choice of acceptance probability, the chain is positive recurrent with a stationary pdf π as described next. The transition kernel for the Metropolis-Hastings algorithm when $\mathbf{x} \neq \mathbf{y}$ is[13]:

$$\begin{aligned} A(\mathbf{x},\mathbf{y}) &= q(\mathbf{x},\mathbf{y})\,\alpha(\mathbf{x},\mathbf{y}) \\ &= q(\mathbf{x},\mathbf{y}) \min\left(1, \frac{\pi(\mathbf{y})\,q(\mathbf{y},\mathbf{x})}{\pi(\mathbf{x})\,q(\mathbf{x},\mathbf{y})}\right), \end{aligned} \tag{17}$$

When $\mathbf{x} \neq \mathbf{y}$, there are two possibilities, namely $\pi(\mathbf{y})\,q(\mathbf{y},\mathbf{x}) \geq \pi(\mathbf{x})\,q(\mathbf{x},\mathbf{y})$ and $\pi(\mathbf{y})\,q(\mathbf{y},\mathbf{x}) < \pi(\mathbf{x})\,q(\mathbf{x},\mathbf{y})$. For brevity, only the first case will be considered for the proof (second case can be proved similarly, but omitted here). When

$\pi(\mathbf{y})\,q(\mathbf{y},\mathbf{x}) \geq \pi(\mathbf{x})\,q(\mathbf{x},\mathbf{y})$, we have[13]

$$\begin{aligned}
\int \pi(\mathbf{x})\,A(\mathbf{x},\mathbf{y})\,d\mathbf{x} &= \int \pi(\mathbf{x})\,q(\mathbf{x},\mathbf{y}) \min\left(1, \frac{\pi(\mathbf{y})\,q(\mathbf{y},\mathbf{x})}{\pi(\mathbf{x})\,q(\mathbf{x},\mathbf{y})}\right) \mathrm{d}\mathbf{x} \\
&= \int \pi(\mathbf{x})\,q(\mathbf{x},\mathbf{y})\,\mathrm{d}\mathbf{x} \\
&= \int \pi(\mathbf{y})\,q(\mathbf{y},\mathbf{x}) \min\left(1, \frac{\pi(\mathbf{x})\,q(\mathbf{x},\mathbf{y})}{\pi(\mathbf{y})\,q(\mathbf{y},\mathbf{x})}\right) \mathrm{d}\mathbf{x} \\
&= \pi(\mathbf{y}) \int q(\mathbf{y},\mathbf{x}) \min\left(1, \frac{\pi(\mathbf{x})\,q(\mathbf{x},\mathbf{y})}{\pi(\mathbf{y})\,q(\mathbf{y},\mathbf{x})}\right) \mathrm{d}\mathbf{x} \\
&= \pi(\mathbf{y}) \int A(\mathbf{y},\mathbf{x})\,\mathrm{d}\mathbf{x} \\
&= \pi(\mathbf{y})\,. \qquad (18)
\end{aligned}$$

This fact proves that the acceptance probability $\alpha(\mathbf{x},\mathbf{y})$ in Eq. (7) leads to the stationary pdf π which is the target pdf.

REFERENCES

[1] Zimmerman, N. H., and Weissenburger, J. T., 1964. "Prediction of flutter onset speed based on flight testing at subcritical speeds". *Journal of Aircraft,* ***1***(4), pp. 190–202.

[2] Price, S. J., and Lee, B. H. K., 1993. "Evaluation and extension of the flutter-margin method for flight flutter prediction". *Journal of Aircraft,* ***30***(3), pp. 395–401.

[3] Poirel, D., Dunn, S., and Porter, J., 2005. "Flutter-margin method accounting for modal parameter uncertainties". *Journal of Aircraft,* ***42***(5), pp. 1236–1243.

[4] Heeg, J., 2007. "Stochastic characterization of flutter using historical wind tunnel data". In Proceedings of the 48th AIAA/ASME/ASCE/AHS/ASC Structures, Structural Dynamics and Materials Conference.

[5] Khalil, M., Poirel, D., Sarkar, A., and Dunn, S., 2009. "Application of bayesian inference to the flutter margin method". In Proceedings of IFASD 2009, the International Forum on Aeroelasticity and Structural Dynamics.

[6] Gilks, W. R., Richardson, S., and Speigelhalter, D. J., eds., 1996. *Markov Chain Monte Carlo in Practice*. Chapman & Hall, London.

[7] Liu, J. S., 2001. *Monte Carlo Strategies in Scientific Computing*. Springer-Verlag, New York.

[8] Tan, S. M., Fox, C., and Nicholls, G., 2009. *Physics 707 (Inverse Problems Course) Lecture Notes*. http://home.comcast.net/∼szemengtan/.

[9] Metropolis, N., Rosenbluth, A. W., Rosenbluth, M. N., Teller, A. H., , and Teller, E., 1953. "Equation of state calculation by fast computing machines". *Journal of Chemical Physics,* ***21***(6), p. 10871092.

[10] Hastings, W. K., 1970. "Monte Carlo sampling methods using Markov chains and their applications". *Biometrika,* ***57***(1), pp. 97–109.

[11] Hoen, C., 2005. "An engineering interpretation of the complex eigensolution of linear dynamic systems". In Proceedings of IMAC XXIII, the 23rd Conference and Exposition on Structural Dynamics.

[12] *Message Passing Interface*. http://www.mpi-forum.org/.

[13] Draper, D., 2003. *Engineering 206 (Bayesian Statistics Course) Lecture Notes*. http://www.soe.ucsc.edu/classes/ams206/Winter03/.