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# FLUID AND STRUCTURE INTERACTION IN COCHLEA'S SIMILAR GEOMETRY 

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#### Abstract

A non linear mathematical model addressing the passive mechanism of the cochlea is proposed in this work. In this respect, the interaction between the basilar membrane seen as an elastic solid and fluids in both scala vestibuli and tympani is developed. Via the fluid/solid interface, a full fluid/solid interaction is taking into account. Furthermore a significant improvement of the existing models has been made in both fluid flow modelling and solid modelling. In the present paper, the flow is three dimensional and the solid is non homogeneous two dimensional membrane where the material parameters depend only on the axial distance. The problem formulation leads to a system of non linear partial differential equations. Solution of the linearized system of partial differential equations of the proposed approach is presented. The numerical results obvious a lower and upper limits of the cochlea resonance frequency versus the material parameters of the basilar membrane. It is shown that a monochromatic acoustic wave energises only a portion of the basilar membrane and the location of the excited portion depends on the frequency of the incident acoustic wave. Those results explain the ability of the cochlea in deciphering the frequency of sound with high resolution in striking similarity with the known experimental results. The mathematical model shows that the excited strip of the basilar membrane by a monochromatic acoustic wave is very small when a transverse wave exists in the basilar membrane. Thus, a transverse wave improves highly the resolution of the cochlea in deciphering the high frequency of the incident acoustic wave.


## 1 Introduction

The ear is an advanced and very sensitive organ of the human body. One of the major tasks of the ear is to detect and analyze noises by transduction. The human auditive system is divided into three different parts as it is shown in figure 1, namely, the outer-, middle- and inner- ear. The sound vibrations are propagated in the external auditory canal and put the tympanic membrane in vibration. The vibration of this membrane is propagated through the middle ear by putting moving the ossicles until at the oval window of the cochlea, entry of inner ear. The movement of this window puts in vibration cochlear fluids which stimulate sensory cells. Our study concerns to the mechanical behavior of the inner ear, and more precisely, the passive mechanism of the cochlea.

The cochlea is a spiraled, hollow, conical chamber of bone, divided into three parts. First, the scala vestibuli (containing fluid named perilymph) lies superior to the cochlea duct and abuts the oval window. Secondly, the scala tympani (containing fluid named perilymph) lies inferior to the scala media and terminates at the round window. Finally, the scala media (containing fluid named endolymph) is the membraneous cochlea duct containing another structure called the organ of Corti. These three parts are separated by two membranes. The Reissner's membrane separates the scala vestibuli from the scala media and the basilar membrane separates the scala media from the scala tympani. The location where the scala tympani and the scala vestibuli merge is called the helicotrema. About the organ of Corti, it's a cellular layer sitting on top of the basilar membrane. It is lined with hair


FIGURE 1. Ear anatomy
cells sensory cells topped with hair-like structures called stereocilia. The cilia of the hair cells tied up to another membrane called the tectorial membrane. When the hair cells are excited by vibration, a nerve impulse is generated in the auditory nerve. These impulses are then sent to the brain. Some experimental and theoretical evidences support the fact that the hair cells might amplify some frequency and compress others. Nonetheless the present work aims to understand the passive mechanism of the cochlea, first because it is not fully understood, and secondly, to emphasize that its contribution is significant on the amplification phenomena. Indeed, following Shera et al. [1], form, amplification and speed propagation of the traveling wave in the cochlea, generated by stapes motion, are not yet understood. An unclosed debate questionnes : does the basilar membrane amplify actively the propagating wave in the cochlea ? Or is the observed amplification right one of the unknown characteristic of the cochlea mechanism ? As the system is subjected to external forces, examining the stability of the passive mechanism could give insight on a possible energy gain or loss of the system, without any work supposed to be done by the basilar membrane.

In order to predict the form of traveling wave in the cochlea by experimental technique, a similarity hypothesis is often needed to convert the local response of the basilar membrane versus time to a global basilar membrane response (that is, amplitude and phase of the basilar membrane velocity (or displacement) versus the cochlea's axis). All these informations could be provided easily by a mathematical model, once it has been val-
idated using published experimental data. Starting from experimental measurements reported by Shera [2], in particular figure 2 in his paper, a qualitative comparisons will be established between experimental and our numerical simulations.

Up to now, different studies have been performed to account for passive mechanism of cochlea.
T. Lin and J. J. Guinan (2004) have used joint timefrequency distribution to analysis click response of cat single auditory-nerve fibers and published measurements of chinchilla basilar-membrane motion. At a single location of the basilar membrane, the energy was found to be distributed over a range of frequencies around the characteristic frequency. To explain the waxing and waning features of the click response of single auditory-nerve fibers, they supposed (i) the existence of more than one resonance frequency representing waves having different wave speeds at a single cochlear location, or, (ii) the existence of reflected wave. The present paper gives strong support to the former hypothesis. Indeed, the proposed model shows that, at a single cochlear location, there are eigenvectors representing waves having different velocity group. Thus the revealed glide experimental irregularities could be explained by the existence of several resonance frequencies at given cochlea's cross section.

In another way, De Boer et al. [3, 4] have modeled the passive mechanism of the cochlea using a linearized two dimensional Euler equation. In this model, domain of channels was supposed to be unbounded in the cochlea axial direction. In this way, introducing an unknown impedance, the fluid motion has been taking into account to give away the basilar membrane response. Resting on this approach, different drawbacks appear. First, no equation has been formulated for the impedance. Secondly, the problem remains open. And finally, though the inverse problem formulation is very useful, it is inadequate for the prediction of cochlea's passive mechanism features. On the contrary, the proposed model in this article is able to take notice of (i) the three dimensional flow, (ii) a full coupling between the membrane and the flow and (iii) the bounded domain in the axial direction.

The passive dynamics of the cochlea was described by Nobili et al. $[6,7]$ using an integral equation. In this study, a qualitative agreement with experimental measurement has been obtained. Moreover, they shown that the model predicts some working cochlea properties providing a force term that accounts for hair cell motility. In this model, the basilar membrane displacement is supposed to be constant in the cochlea's cross section and a spring-like structure is used to describe the membrane motion. A more realistic description of the cochlea behavior needs to take into account the basilar membrane's transversal displacement. This is performed with the proposed model in this article. Furthermore, this proposed model predicts mechanical features of the cochlea without including a force term.

Another spring-like structure model has been used for the basilar membrane in Zweing et al. [8]. In their approach, the
pressure difference across the scala media and the volume velocity of the fluid flow down the cochlea is assumed to satisfy a transmission-line equation. The model yields one dimension wave equation with varying mechanical properties. The WKB method ${ }^{1}$ as well as finite difference method are used to solve the obtained wave equation. A transfer function, defined as the ratio of the basilar membrane velocity and the round window velocity, was shown to be in good agreement with its counter part obtained by experimental measurements.

In the present paper, in order to simplify the mathematical description of the cochlea mechanical behavior, it was considered as a strait stretched chamber rather than conical one. However, the proposed formulation is independent of the cross section form, as well as of its lumen. Moreover, a decreasing lumen toward the apex could be considered. We neglect also the variation of the cochlea's cross section, this is of course a severe approximation, nevertheless we think that such approximation is need in order to have an insight on the cochlea mechanism without tackling a cumbersome equations. The basilar membrane, Reissner's membrane, the scala media and the organ of Corti are considered as a single elastic membrane. The fluid in both scala vestibuli and tympani channels are non viscous and we assume an irrotational flow. Although the numerical treatment of non linear equations could be attemped, the aim of the present work is the analysis of linearized equations. As it will be presented, this analysis gives interesting informations for the understood of the cochlea's passive mechanism. The main hypothesis of the proposed model rests on the axial flow velocity : it is supposed constant in the cochlea's cross section. This hypothesis is justified because the boundary layer has no time to grow and pervade the section.

The paper is divided in five sections: the first and second ones are devoted to the mechanical formulation of the problem and the flow modelling. The third part focuses on the linear model. The fourth one proposes an asymptotic analysis. Finally, numerical results and qualitative comparison with experimental data are presented. In the last part, conclusions are discussed.

## 2 Formulation of the cochlear problem

Hereafter, our approach is presented. First, assumptions of the cochlear mechanical model are listed. Then the fluid modelling and the solid one are described. Finally the proposed methodology to solve the problem including a full fluid/structure coupling is detailled.

### 2.1 Assumptions of the cochlear mechanical model.

In order to simplify and eliminate technical complications on the mathematical description of the cochlea mechanical behavior, we make several simplifying assumptions.

[^0]First, we work with simplest geometry, considering the cochlea as a plan rather than spiral chamber though the present formulation remains valid for an arbitrary cochlea's cross section shape. We assume also incompressible and non viscous scala fluids (vestibuli and tympani), and irrotational flow. A same fluid is considered for the differents channels which implies that the fluid mass density in the vestibuli and tympani channels are the same, written $\rho_{f}$. Reissner and basilar membranes, the scala media and the organ of Corti are considered as a single elastic membrane, that shall be called afterwards the basilar membrane (BM). That is, we consider two channels of length $L$, separated by a partition as it is shown in figure.

Except at the two windows (oval and round), we assume that the box is surrounded by rigid bone so that no fluid enters or leaves the cochlea. The fluid's volume is displaced by the two windows; windows's displacement are therefore equal and opposite in direction.

The partition is composed with a rigid part and a moving one, representing corticoi bone and BM respectively. The $x$ axis extends longitudinally from the base (at $x=0$ ) to the apex (at $x=L$ ), and the $z$ axis is oriented perpendicular to the BM, which, when the BM at rest, is located in the $x y$ plane. The BM width is $r(x)$.

About the helicotrema, we assume its location at $x=L$. A single point is considered to be representative of the exchange between the vestibular and tympani channels. Then, $L$ is also the BM length and specific boundary conditions in this point will be described. In particular, velocity is null in both vestibuli and tympani channels. The continuity of the velocity througth the BM at $x=L$ is reinforced, that is, the BM at $x=L$ is at rest.

Let $W(x, z, t)$ be the BM's displacement in $z$ direction and $V_{0}$ the velocity of the membrane of the oval window. We assume that the BM's displacement in $x$ and $y$ directions are small in comparison with $W(x, y, t)$.

Let now present the basic equations of the cochlear mechanical problem.

### 2.2 Basic equations of the cochlear mechanical problem.

Because of irrotational flow assumption, the velocities in the two channels are the gradient of two scalar functions, $\Phi_{V}$ and $\Phi_{T}$, respectively for vestibuli and tympani channels, which satisfy the following Laplace equations :

$$
\begin{equation*}
\Delta \Phi_{V}=0 ; \quad \Delta \Phi_{T}=0 \tag{1}
\end{equation*}
$$

$$
\begin{equation*}
V^{V}=\nabla \Phi_{V} ; \quad V^{T}=\nabla \Phi_{T} \tag{2}
\end{equation*}
$$

subjected to suitable boundary conditions, namely, $\nabla \Phi_{V}=0$ at rigid part of the cochlea and $\nabla \Phi_{V}$ equals the velocity of the moving part of the cochlea and a similar boundary condition should be imposed on $\Phi_{T}$.

Taking account of our assumptions, we can establish the boundary conditions associated with precedent Laplace equations (1) for the two channels. Since the scalae walls are assumed rigid, the corresponding normal component of the fluid velocity must vanish. Just above the cochlear partition, the fluid velocity must equal the velocity of the MB , i.e. $V_{\text {Fluid }}=V_{B M}$,

$$
\begin{equation*}
\frac{\partial \Phi_{V}}{\partial z}=\frac{\partial W}{\partial t} ; \quad \frac{\partial \Phi_{T}}{\partial z}=\frac{\partial W}{\partial t} \tag{3}
\end{equation*}
$$

Just inside the oval window, the $x$ component of the fluid velocity must equal the velocity of the stapes.

At the end of the cochlea, we assume that the velocity is null in both vestibuli and tympani channels and that continuity is ensure througth the BM (at the helicotrema location). Finally, through the partition's bone, the velocity is null. Pressures in vestibuli and tympani channels, $p_{V}$ and $p_{T}$ respectively, could be obtained by generalised Bernoulli theory [11], thus they verify the following relations :

$$
\begin{align*}
p_{V} & =-\rho_{f} \frac{\partial \Phi_{V}}{\partial t}-\frac{\rho_{f}}{2}\left(\nabla \Phi_{V}\right)^{2}+C(t) \\
p_{T} & =-\rho_{f} \frac{\partial \Phi_{T}}{\partial t}-\frac{\rho_{f}}{2}\left(\nabla \Phi_{T}\right)^{2}+C(t) \tag{4}
\end{align*}
$$

where $\nabla$ stands for the gradient operator, $\rho_{f}$ denotes the fluid mass density (volumic) in the vestibuli and tympani channels, and $C(t)$ is an arbitrary constant value. Then, the force applied onto the BM by the pressure by surface unity is

$$
\begin{align*}
& q=\left(p_{T}-p_{V}\right)=\rho_{f} \frac{\partial\left(\Phi_{V}-\Phi_{T}\right)}{\partial t} \\
&+\rho_{f} \frac{1}{2}\left[\left(\nabla \Phi_{V}\right)^{2}-\left(\nabla \Phi_{T}\right)^{2}\right] \tag{5}
\end{align*}
$$

Taking account of our assumptions, the BM is assimilated to a two dimensional membrane where the displacement of the material point located at $W(x, z, t)$ satisfies the following equation

$$
\begin{equation*}
\rho_{s} \frac{\partial^{2} W}{\partial t^{2}}-\frac{\partial}{\partial x}\left(T_{x} \frac{\partial W}{\partial x}\right)-\frac{\partial}{\partial y}\left(T_{y} \frac{\partial W}{\partial y}\right)=+q \tag{6}
\end{equation*}
$$

In the precedent equation, $\rho_{s}$ stands for the surface mean mass density of the Reissner's membrane, basilar membrane, the scala
media and the organ of Corti. The coefficients $T_{x}$ and $T_{y}$ are axial and transversal tensions of the membrane. They are supposed to be dependent on the axial distance $x$. The force term $q$ is the pressure applied by the flow in the scala vestibuli and the scala tympani on the BM which will be estimated in the next section.

The problem is now to solve equation (1) for the two scalar functions, $\Phi_{V}$ and $\Phi_{T}$, subject to the prescribed boundary conditions in order to determine the passive mechanical behavior of the cochlea. As we can see, the boundary conditions involve the BM's displacement. Then, the unknows are not only the scalar functions, $\Phi_{V}$ and $\Phi_{T}$ describing what occur in fluids, but also $W(x, y, t)$ the BM's displacement, along time and space. Note that others authors have been interested by this problem. Shera et al. [10] have proposed a 1D- and a 2D- models, but their solving was achieved by considering the average pressure on each scala. This point of view reduce the problem to one dimension. In this paper, the proposed approach, presented in the next section, does not need to consider the average pressure.

### 2.3 Flow modelling

In order to solve the BM's displacement using the system of equations described in the precedent section, it necessary to inverse Laplace operator in a domain with moving boundary. This is a challenging task because the computation grid need to be up dated at each time step in order to take account of the BM's displacement. In this section, we present a simplified formulation permitting to avoid up dating the computation grid and leading to a tractable model, removing the need to inverse Laplace operator. It's simplified because based on the main hypothesis detailled previously in section 2.1 , that the axial velocity component, $V_{x}^{T}$ and $V_{x}^{V}$, are constant in the cross section of the vestibuli and tympani channels respectively ${ }^{2}$. This approximation is valid if the BM's displacement is small in comparison with the radius of the cochlea. In the present approach, there is no need to neglect the fluid's velocity component in $z$ direction, $V_{z}$, in comparison to $V_{x}$ and $V_{y}$. As it shall be seen, increasing the order of the differential equations governing the BM removes the need to solve for three dimensional flow, without simplifying assumptions. This technique leads to a lost of some informations which will appear later, in the form of some arbitraryness on the boundary conditions used to solve the BM's equation. The correct treatment of the boundary conditions is possible but relegated to another work.

The first point of the proposed approach so as to solve the problem is to establish the mass balance in the domain included between two cross sections separated by a gap $d x$. Let $S_{V}^{0}$ and

[^1]$S_{T}^{0}$ be the respective cross section's area of the vestibuli and tympani channels when the BM is at rest. When the BM moves, the surfaces of the scala vestibuli and scala tympani at the position $x$ become
\[

$$
\begin{align*}
& S_{V}(x, t)=S_{V}^{0}-\int_{-r(x)}^{r(x)} W(x, y, t) d y \\
& S_{T}(x, t)=S_{T}^{0}+\int_{-r(x)}^{r(x)} W(x, y, t) d y \tag{7}
\end{align*}
$$
\]

Here $r(x)$ corresponds to the BM's width, varying along $x$ axis. Thus, in the scala vestibuli, the difference between two sections located at $x$ and $x+d x$ are expressed by :

$$
\begin{array}{r}
S_{V}(x+d x, t)-S_{V}(x, t)=-\int_{-r(x)}^{r(x+d x)} W(x+d x, y, t) d y \\
+\int_{-r(x)}^{r(x)} W(x, y, t) d y \tag{8}
\end{array}
$$

When $d x$ approches zero and assuming $r(x)$ varies slowly along $x$ axis (i.e. the term involving $\frac{d r}{d x}$ is neglected), we obtain:

$$
\begin{equation*}
S_{V}(x+d x, t)-S_{V}(x, t)=-d x \int_{-r(x)}^{r(x)} \frac{\partial W(x, y, t)}{\partial x} d y \tag{9}
\end{equation*}
$$

A similar relation is established in the scala tympani :

$$
\begin{equation*}
S_{T}(x+d x, t)-S_{T}(x, t)=d x \int_{-r(x)}^{r(x)} \frac{\partial W(x, y, t)}{\partial x} d y \tag{10}
\end{equation*}
$$

Starting from these relations, we establish the mass balance in the domain enclosed by the surface $S$ defined by the following boundaries: $S_{V}(x, t), S_{V}(x+d x, t)$, the partition surface and the rigid walls. It gives :

$$
\begin{array}{r}
\int_{S} \vec{V} \cdot \vec{n} d S=0 \Leftrightarrow-\int_{S_{V}(x, t)} V_{x}(x) d S \\
+\int_{S_{V}(x+d x, t)} V_{x}(x+d x) d S+\int_{\Delta S_{V}^{\text {lateral }}} \vec{V} \cdot \vec{n} d S=0 \tag{11}
\end{array}
$$

As we supposed that $V_{x}(x, t)$ is constant in the channel's cross section, the precedent equation could be simplified by :

$$
\begin{align*}
& -V_{x}(x) S_{V}(x, t)+V_{x}(x+d x) S_{V}(x+d x, t) \\
& \quad-\int_{x}^{x+d x} \int_{-r(x)}^{r(x)} \frac{\partial W(x, y, t)}{\partial t} d y d x=0 . \tag{12}
\end{align*}
$$

Using equations (8), we obtain

$$
\begin{align*}
& V_{x}(x+d x)\left[S_{V}(x, t) d x-\int_{-r(x)}^{r(x)} \frac{\partial W(x, y, t)}{\partial x} d y\right] \\
& -V_{x}(x) S_{V}(x, t)-\int_{x}^{x+d x} \int_{-r(x)}^{r(x)} \frac{\partial W(x, y, t)}{\partial t} d y d x=0 \tag{13}
\end{align*}
$$

which can be written

$$
\begin{array}{r}
{\left[V_{x}(x+d x)-V_{x}(x)\right] S_{V}(x, t) d x} \\
-V_{x}(x+d x) \int_{-r(x)}^{r(x)} \frac{\partial W(x, y, t)}{\partial x} d y \\
-\int_{x}^{x+d x} \int_{-r(x)}^{r(x)} \frac{\partial W(x, y, t)}{\partial t} d y d x=0 \tag{14}
\end{array}
$$

When $d x$ approches zero, we obtain

$$
\begin{align*}
\frac{\partial V_{x}}{\partial x} S_{V}(x, t) & -V_{x}(x) \int_{-r(x)}^{r(x)} \frac{\partial W(x, y, t)}{\partial x} d y \\
& -\int_{-r(x)}^{r(x)} \frac{\partial W(x, y, t)}{\partial t} d y=0 \tag{15}
\end{align*}
$$

As in this case $V_{x}=\frac{\partial \Phi_{V}}{\partial x}$, the precedent equation becomes

$$
\begin{align*}
\frac{\partial^{2} \Phi_{V}}{\partial x^{2}} S_{V}(x, t) & -\frac{\partial \Phi_{V}}{\partial x} \int_{-r(x)}^{r(x)} \frac{\partial W(x, y, t)}{\partial x} d y \\
& -\int_{-r(x)}^{r(x)} \frac{\partial W(x, y, t)}{\partial t} d y=0 \tag{16}
\end{align*}
$$

Using equation (7) and eliminating $S_{V}(x, t)$ leads to the following equation

$$
\begin{array}{r}
\frac{\partial^{2} \Phi_{V}}{\partial x^{2}}\left[S_{V}^{0}-\int_{-r(x)}^{r(x)} W(x, y, t) d y\right] \\
-\frac{\partial \Phi_{V}}{\partial x} \int_{-r(x)}^{r(x)} \frac{\partial W(x, y, t)}{\partial x} d y-\int_{-r(x)}^{r(x)} \frac{\partial W(x, y, t)}{\partial t} d y=0 . \tag{17}
\end{array}
$$

In a similar way, the fluid in the scala tympani verifies the following equation

$$
\begin{array}{r}
\frac{\partial^{2} \Phi_{T}}{\partial x^{2}}\left[S_{T}^{0}+\int_{-r(x)}^{r(x)} W(x, y, t) d y\right] \\
+ \\
\quad \frac{\partial \Phi_{T}}{\partial x} \int_{-r(x)}^{r(x)} \frac{\partial W(x, y, t)}{\partial x} d y  \tag{18}\\
\quad+\int_{-r(x)}^{r(x)} \frac{\partial W(x, y, t)}{\partial t} d y=0
\end{array}
$$

Hereinafter we deal with equations (5), (6), (17), (18) instead of inversing Laplace operator. Starting from these equations, an approximate model will be established in next sections. To this end, we introduce the function $\Phi=\Phi_{V}-\Phi_{T}$.

## 3 Linear model

The linearization of precedent equations leads to the first step of model formulation. Neglecting the non linear term in fluid's equations (5), (17) and (18) the basic equations become :

$$
\begin{array}{r}
q=\rho_{f} \frac{\partial}{\partial t} \Phi(x, y, z, t) \\
\frac{\partial^{2}}{\partial x^{2}} \Phi_{V}(x, y, z, t)=\frac{1}{S_{V}^{0}} \int_{-r(x)}^{r(x)} \frac{\partial}{\partial t} W(x, y, t) d y \\
\frac{\partial^{2}}{\partial x^{2}} \Phi_{T}(x, y, z, t)=-\frac{1}{S_{T}^{0}} \int_{-r(x)}^{r(x)} \frac{\partial}{\partial t} W(x, y, t) d y \tag{21}
\end{array}
$$

Assuming that $T_{x}$ and $T_{y}$ depend on $x$ only, the BM's equation (6) become :

$$
\begin{equation*}
\rho_{s} \frac{\partial^{2} W}{\partial t^{2}}-\frac{d T_{x}}{d x} \frac{\partial W}{\partial x}-T_{x} \frac{\partial^{2} W}{\partial x^{2}}-T_{y} \frac{\partial^{2} W}{\partial y^{2}}=\rho_{f} \frac{\partial \Phi}{\partial t} \tag{22}
\end{equation*}
$$

and using the function $\Phi$, the two last equations of (20) can be written differently :

$$
\begin{equation*}
\frac{\partial^{2}}{\partial x^{2}} \Phi(x, y, z, t)=\left(\frac{1}{S_{V}^{0}}+\frac{1}{S_{T}^{0}}\right) \int_{-r(x)}^{r(x)} \frac{\partial W(x, y, t)}{\partial t} d y \tag{23}
\end{equation*}
$$

Note that this equation implies that the second order derivative of $\Phi(x, y, z, t)$ depends only on $x$ and $t$. However the flow could be three-dimensional one. In the present approach, we use this property of the second order derivative of $\Phi(x, y, z, t)$, to avoid solving for three-dimensional flow. To this end, the second order partial derivative of the BM's equation (22) with respect to $x$ has been carried out :

$$
\begin{array}{r}
\rho_{s} \frac{\partial^{4} W}{\partial x^{2} \partial t^{2}}-\frac{\partial^{2}}{\partial x^{2}}\left(\frac{d T_{x}}{d x} \frac{\partial W}{\partial x}+T_{x} \frac{\partial^{2} W}{\partial x^{2}}\right) \\
-\frac{\partial^{2}}{\partial x^{2}}\left(T_{y} \frac{\partial^{2} W}{\partial y^{2}}\right)=\rho_{f} \frac{\partial^{3}}{\partial x^{2} \partial t} \Phi \tag{24}
\end{array}
$$

Then, to obtain an equation involving only $W, \frac{\partial^{3}}{\partial x^{2} \partial t} \Phi$ needs to be removed. Thus, by taking the partial time derivative of equation (23), we obtain:

$$
\begin{equation*}
\frac{\partial^{3}}{\partial t \partial x^{2}} \Phi(x, y, z, t)=\left(\frac{1}{S_{V}^{0}}+\frac{1}{S_{T}^{0}}\right) \int_{-r(x)}^{r(x)} \frac{\partial^{2} W(x, y, t)}{\partial t^{2}} d y \tag{25}
\end{equation*}
$$

Finally, the coupled fluid/solid equation is obtained by eliminating $\Phi$ from the two last equations, leading to:

$$
\begin{array}{r}
\rho_{s} \frac{\partial^{4} W}{\partial x^{2} \partial t^{2}}-\frac{\partial^{2}}{\partial x^{2}}\left(\frac{d T_{x}}{d x} \frac{\partial W}{\partial x}+T_{x} \frac{\partial^{2} W}{\partial x^{2}}\right)-\frac{\partial^{2}}{\partial x^{2}}\left(T_{y} \frac{\partial^{2} W}{\partial y^{2}}\right)= \\
\rho_{f}\left(\frac{1}{S_{V}^{0}}+\frac{1}{S_{T}^{0}}\right) \int_{-r(x)}^{r(x)} \frac{\partial^{2} W(x, y, t)}{\partial t^{2}} d y .
\end{array}
$$

Moreover, we integrate the precedent equation over the range [$\mathrm{r}(\mathrm{x}), \mathrm{r}(\mathrm{x})$ ] (for $y$ parameter) to obtain :

$$
\begin{array}{r}
\int_{-r(x)}^{r(x)}\left[\rho_{s} \frac{\partial^{4} W}{\partial x^{2} \partial t^{2}}-\frac{\partial^{2}}{\partial x^{2}}\left(\frac{d T_{x}}{d x} \frac{\partial W}{\partial x}+T_{x} \frac{\partial^{2} W}{\partial x^{2}}\right)\right. \\
\left.-\frac{\partial^{2}}{\partial x^{2}}\left(T_{y} \frac{\partial^{2} W}{\partial y^{2}}\right)\right] d y= \\
2 r(x) \rho_{f}\left(\frac{1}{S_{V}^{0}}+\frac{1}{S_{T}^{0}}\right) \int_{-r(x)}^{r(x)} \frac{\partial^{2} W(x, y, t)}{\partial t^{2}} d y
\end{array}
$$

A dimensionless coupled fluid/solide equation is considered to obtain numerical solution. In this way, we use dimensionless variables: let $h$ and $\omega_{0}^{-1}$ be the distance and time scales. Then, we define the dimensionless variables as follow:

$$
\begin{array}{r}
x^{*}=\frac{x}{h} ; \quad y^{*}=\frac{y}{h} ; \quad t^{*}=t \omega_{0} \\
r^{*}=\frac{r}{h} ; \quad S_{V / T}^{0 *}=\frac{S_{V / T}^{0}}{h^{2}} \quad ; \quad \tilde{w}^{*}=\frac{\tilde{w}}{h} \\
T_{x}^{*}=\frac{T_{x}}{\rho_{s} h^{2} \omega_{0}^{2}} ; \quad T_{y}^{*}=\frac{T_{y}}{\rho_{s} h^{2} \omega_{0}^{2}} ; \quad \rho^{*}=\frac{h \rho_{f}}{\rho_{s}} \tag{28}
\end{array}
$$

Here and after, we do not keep the star for reading clarity and writing easiness. Further, the BM's motion is supposed sinusoidal in $y$ direction. That is,

$$
\begin{array}{r}
W(x, y, t)=w(x, t) \cos \left(\frac{n \pi y}{2 r(x)}\right) \\
(x, y, t) \in[0, L] \times[-r(x), r(x)] \times[0, T] \tag{29}
\end{array}
$$

The trigonometric function is chosen in order to satisfy the boundary condition imposed by the lateral tethering of the BM. More general solution involving more than one Fourier's components in $y$ direction could be sought and doing so is straight forward. After substitution in the equation (26), we obtain :

$$
\begin{array}{r}
\frac{\partial^{4} w}{\partial x^{2} \partial t^{2}}-\frac{\partial^{2}}{\partial x^{2}}\left(\frac{d T_{x}}{d x} \frac{\partial w}{\partial x}+T_{x} \frac{\partial^{2} w}{\partial x^{2}}\right) \\
+\left(\frac{n \pi}{2 r(x)}\right)^{2} \frac{\partial^{2}}{\partial x^{2}}\left(T_{y} w\right) \tag{30}
\end{array}=2 r(x) \rho\left(\frac{1}{S_{V}^{0}}+\frac{1}{S_{T}^{0}}\right) \frac{\partial^{2} w}{\partial t^{2}} .
$$

In order to obtain a simplified equation, terms involving the derivatives of $r(x)^{3}$ are neglected. For the numerical simulation, we shall consider a harmonic mode for $w$, in the form :

$$
\begin{equation*}
w(x, t)=\tilde{w}(x) e^{i \omega t} \tag{31}
\end{equation*}
$$

Thus, $\tilde{w}$ obeys the following equation :

$$
\begin{array}{r}
\omega^{2} \frac{d^{2} \tilde{w}}{d x^{2}}+\frac{d^{2}}{d x^{2}}\left[\frac{d T_{x}}{d x} \frac{d \tilde{w}}{d x}+T_{x} \frac{d^{2} \tilde{w}}{d x^{2}}\right] \\
-\left(\frac{n \pi}{2 r(x)}\right)^{2} \frac{d^{2}}{d x^{2}}\left[T_{y} \tilde{w}\right]=2 r(x) \rho\left(\frac{1}{S_{V}^{0}}+\frac{1}{S_{T}^{0}}\right) \omega^{2} \tilde{w} \tag{32}
\end{array}
$$

associated with the boundary conditions at $x=0$ and at $x=L$

$$
\begin{equation*}
\tilde{w}=0, \frac{\partial \tilde{w}}{\partial x}=\text { constant } \tag{33}
\end{equation*}
$$

Here the constant involved in the boundary condition is arbitrary. An exact treatment of the boundary conditions involving the oval window velocity is possible though is delayed to an other work.As we solve a linear problem, the amplitude of the displacement in a given location is proportional to this constant. The value of the constant is chosen in such a way the membrane displacement never exceeds the width of the cochlea cross section. The precedent equation shall be solved numerically. Starting from this, we establish an asymptotic analysis in the next section.

## 4 Asymptotic analysis

In this section we limit the analysis to the short waves. In this way, we suppose that the rheological and geometrical parameters of the cochlea vary slowly with $x$ in comparison with the variation of BM's displacement. Thus, the analysis is valid when

$$
\begin{equation*}
\left|\frac{d T_{x}}{d x}\right| \ll\left|\frac{\partial w}{\partial x}\right| \tag{34}
\end{equation*}
$$

A similar relations are hold for $T_{y}, S_{v}, S_{t}$ and $r(x)$. Precedent hypotheses allows to perform a local analysis. Moreover the dependence on $x$ of axial and transversal tensions, is as follows

$$
\begin{array}{r}
T_{x}=A_{1}+B_{1} e^{-\Gamma_{1} x} \quad ; T_{y}=A_{2}+B_{2} e^{-\Gamma_{2} x} \\
r(x)=r_{L} \frac{x}{L}+r_{0} \tag{35}
\end{array}
$$

[^2]where $A_{1}, B_{1}, A_{2}, B_{2}, \Gamma_{1}$ and $\Gamma_{2}$ are constants. Then, a solution of the basic equation in the form normal mode has been sought, namely
\[

$$
\begin{equation*}
w(x, t)=\tilde{\tilde{w}} e^{i k x+i \omega t} \tag{36}
\end{equation*}
$$

\]

Hence the dispersion's equation associated with equation (30) is established :

$$
\begin{array}{r}
{\left[k^{2}+2 r(x) \rho\left(\frac{1}{S_{V}^{0}}+\frac{1}{S_{T}^{0}}\right)\right] \omega^{2}} \\
+\left[-i k \frac{d^{3} T_{x}}{d x^{3}}+3 k^{2} \frac{d^{2} T_{x}}{d x^{2}}+3 i k^{3} \frac{d T_{x}}{d x}-k^{4} T_{x}\right] \\
+\left(\frac{n \pi}{2 r(x)}\right)^{2}\left[\frac{d^{2} T_{y}}{d x^{2}}+2 i k \frac{d T_{y}}{d x}-k^{2} T_{y}\right]=0 . \tag{37}
\end{array}
$$

Using equations (30) and (37), some numerical results in the form of dimensionless quantities will be presented and commented in the next section.

## 5 Results

In precedent section, a mathematical model for the cochlea's passive mechanism was established with a full interaction between the flow in the channels and the BM. The flow in both vestibuli and tympani channels is three dimensional, non viscous and with small velocity amplitude. Direct description of the three dimensional flow is avoided by increasing the order of the differential equation describing the cochlea's passive mechanism. The BM, the organ of Corti and the scala media were supposed to behave as a membrane with varying rheological and geometrical properties. The formulation led to a non linear forth order partial differential equation on space and second order on time. Coefficients of this partial differential equation depend on the axial distance. A linearized version of the dynamic equation was obtained. In the linear version of the model, the dependence on time was supposed to be in the form of a normal mode, i.e. of the form $e^{i \omega t}$. The displacement in the transversal direction was supposed to be sinusoidal with arbitrary transversal wave number $n$. Then, a finite difference method, with uniform grid, is used to solve the resulting dimensionless linear equation, namely equation (30).

All the numerical results shown here are obtained with 200 points located in the axial direction and separated by constant increment. The aim of the numerical computation is to know if the model could predict the main features of the cochlea's passive mechanism, that is, the place to frequency relationship, the upper and lower hearing limites. In this way, two numerical simulations have been performed. In the first one, $\omega$ is considered as a real number and in the second one, $\omega$ is a complex number.

Geometrical parameters used are $L=15, S_{V}^{0}=0.5, S_{T}^{0}=0.5$, $r_{0}=0.2$ and $r_{L}=1$. Material parameters used are $A_{1}=A_{2}=10$, $B_{1}=B_{2}=0.01, \Gamma_{1}=\Gamma_{2}=1$ for $T_{x}$ and $T_{y}$, and $\rho=\frac{h \rho_{f}}{\rho_{s}}=1$.

For the first numerical simulation, results are shown in figures 3, 4, 5. In figures 3 and 4, the BM's displacement is plotted versus the dimensionless axial distance of the cochlea for some chosen dimensionless real frequencies $\frac{\omega}{\omega_{0}}, \omega_{0}^{-1}$ being the time scale. Subfigures $3 \mathrm{a}, 3 \mathrm{~b}, 3 \mathrm{c}$ and $4 \mathrm{a}, 4 \mathrm{~b}, 4 \mathrm{c}$ relate to the BM's displacement for different transversal wave numbers ( $n=1 \ldots 6$ ). The curves in each subfigure are obtained for various decreasing values of $\frac{\omega}{\omega_{0}}$. For all transversal wave number, the maximum of the BM's displacement shiftes continuously toward the apex when the stimulus frequency, $\frac{\omega}{\omega_{0}}$, decreases. On the one hand, for $n=1$ and high frequency stimulus, the BM is fully displaced, while for low frequency stimulus the displacement concerns only the downstream portion of the BM. On the other hand, for other transversal wave numbers ( $n \geq 2$ ), a portion of the BM located at the base remains quiet, the higher the frequency, the longer the portion and the higher the transversal wave number, the longer the portion. Note that we did not find a solution outside the range of the frequency indicated in the figure in surprising agreement with upper and lower frequency limits ability of the cochlea and consequently our ability of hearing the very low and very high frequency. The fact that the BM's displacement observed in figures 3 and 4, involves a large portion of the BM is not what one could expect in view of the sharp frequency selectivity of the cochlea and its ability in decoding two close frequencies. Of course, it is tempting to use a more general approach in which $\frac{\omega}{\omega_{0}}$ could be considered complex number rather than real one in order to see if a possible more sharp resolution by the cochlea could be obtained.

In order to elevate the arbitrariness in the choice of $\frac{\omega}{\omega_{0}}$, we perform an asymptotic analysis (local analysis) determining eigenvalue of the precedent system of equations. The local analysis is valid for high wave number (short wave). In this analysis we suppose that the variation of the rheological and geometrical cochlea's parameters are slow in comparison with the variation of the BM's displacement with respect to axial distance. Thus, a solution in the normal mode form could be sought leading to a second order polynomial equation on $\frac{\omega}{\omega_{0}}$.

Figure 5 shows the unstable mode in (frequency, amplification rate)-plane, that is, the amplification rate versus the frequency when the wave number varies. Curves relate to different axial positions from the base to the apex of the BM. Figure 5a illustrates the unstable mode. It differs on by algebraic signe from the stable one. It was found that: (i) the system is unstable, (ii) the amplification rate is higher near the base and decreases when axial distance increases. It reaches its minimum value at the apex. In figure 5b, the group velocity related to the unstable mode is plotted versus the (axial) wave number $k$. Notice that the group speed of the unstable mode is negative except for a
short range of low wave number, where the analysis is less valid. Therefore, the unstable mode propagates toward the oval window. Consequently the well known otoacoustic emission phenomena could be a consequence of this unstable mode. Indeed, if we suppose that the stable mode is dampened quickly enough, we could conclude that there's no reflection coming from the cochlea's end, and then, otoacoutic emissions are due to the unstable mode's propagation in upstream direction.

For the second numerical simulation, results are shown in figures $6,7,8$. Guided by the eigenvalue obtained by the local analysis, we carried out a numerical computation with complex values of $\frac{\omega}{\omega_{0}}$. As suggested by the local analysis, we take $\frac{\omega}{\omega_{0}}=\operatorname{Re}\left\{\frac{\omega}{\omega_{0}}\right\}\left(1+\frac{1}{2} i\right)$, where $i=\sqrt{-1}$. Then, the linear equation is solved by finite difference method, for a set of values of $\frac{\omega}{\omega_{0}}$. Figure 6 shows the envelope of the BM's displacement, for different transversal wave number. The transversal wave number $n$ is indicated on the top of each curve. The envelope describes the maximum value of the BM's displacement versus value of $\operatorname{Re}\left(\frac{\omega}{\omega_{0}}\right)$. For $n=1$, the envelope curve rises sharply at $\operatorname{Re}\left\{\frac{\omega}{\omega_{0}}\right\} \approx 5$ and falls sharply at $\operatorname{Re}\left\{\frac{\omega}{\omega_{0}}\right\} \approx 40$. Outside of this dimensionless frequency range, namely [5,40], the numerical method used here gives only a solution of very small BM's displacement amplitude. Thus, to move significantly the BM with a frequency stimulus outside this range, the ear needs a very loud sound. It could be observed that (i) the BM is excited with a relatively small frequency range when $n>1$, and, (ii) this range is shifted to lower frequency when $n$ increases. Note that the BM's displacement obtained by the present analysis bears a striking qualitative agreement with the experimental displacement showing in figure 2 panel (c) in the paper Shera et al. [1].

In subfigures 7a, 7b, 7c , the BM's displacement is plotted for three different complex frequencies and for the first transversal wave number $(n=1)$. These plots illustrate that : (i) for high frequency, the BM's displacement is located near the base (figure 7 a), (ii) for middle frequency, the displacement is located in the middle of the BM (figure 7b), and finally, (iii) for low frequency, the displacement is near the apex (figure 7c). These results point out the ability of the proposed mathematical model to predict the place to frequency relationship.

The last figures 8 and 9, show the BM's displacement induced by modes with higher transversal wave number, $n>1$. The envelope shown in figure 6 is used to select the frequency for which we compute the BM's displacement for $n>1$. Higher the transversal wave number, closer the BM's displacement to the apex. It worthy to notice that, when $n>1$, the BM's displacement is very sharp and involves only a small part of its whole length. this sharp displacement could be at the root of the cochlea's ability of distinguish too close frequency

## 6 Conclusion

A mathematical model for the cochlea's passive mechanism has been established in this article. A full interaction between the flow in scala channels and the basilar membrane is allowed. The three dimensional flow in both vestibuli and tympani scalae are irrotational and fluids are non viscous. The main hypothesis of the proposed model rests on the axial flow velocity, supposed to be constant. The scala media, the tectorial membrane, the organ of Corti and the basilar membrane have been considered together as a stretched membrane called, for abbreviation, basilar membrane. To avoid solving three dimension Laplace equation for the determination of the flow potential, a mass balance was used leading to basic equations describing fluid's motion. Then, the potential flow was eliminated by cross derivation. In this article, in order to simplify the mathematical description of the cochlea's mechanical behavior, it was considered as a strait stretched chamber rather than conical one. However, the proposed formulation is independent of the cross section's form as well as of its lumen. An asymptotic analysis of the estabilished equation was performed for short waves. It was found that the system has two complex eigenvalues and one of them represent an unstable mode. Therefore, as the system is subjected to external forces, in the circumstance, the incoming acoustic wave, the energy gain observed in experimental work could be explained by the instability of the system rather than by a possible work done by the outer hair cells.

A finite difference method has been used to solve the founded equations for two different types of dimensionless frequencies. In the first numerical simulation, the underlined frequency is real, while in the second one, the frequency is complex.

For the first one, the BM's displacement was plotted versus the axial distance. When the frequency is real, no sharp BM's displacement has been found, the displacement concerns all the basilar membrane instead, particularly for high frequency. Nevertheless, the maximum of the BM's displacement moves from the base toward the apex when the frequency decreases. For the transversal wave number higher than one, a portion of the basilar membrane near the base remain at rest. The length of this portion increases when the transversal wave number increases and when the stimulus frequency decreases.

In the second numerical simulation, we considered a complex frequency. It was found that the BM's displacement depends strongly on the amplitude of the complex frequency. In that, only a short portion of the basilar membrane is set in motion for a given frequency and a given transversal wave number. For high frequency, the displacement concerns a portion of the basilar membrane located near the base which moves toward the apex when the amplitude of the complex frequency decreases. These results show that the place-to-frequency relation is an intrinsic property of the cochlea's passive mechanism. Moreover, the obtained displacement shows a qualitative agreement with the
displacement obtained by the experience, Figure 2 pannel C in the paper of Shera et al. [1]. It was found that the BM's displacement is significant only in a range of frequency. Thus, (i) the BM's displacement becomes significant only above some critical lower limit of low frequency, then, (ii) it rises very sharply, (iii) to finally fall very sharply above some critical upper limit of high frequency. There's a striking similarity with the well known upper and lower frequency's limits of hearing.

In conclusion, this study is a significant improvement to the understanding of the cochlea's passive mechanism, supplying lower and upper limits of the cochlea's resonance frequency, versus the rheological parameters of the basilar membrane. Moreover, the BM's displacement obtained by the present model shows a striking similarity with the displacement obtained by experimental measurements. The next steps will be devoted to improve the present model taking into account (i) new BM's models, (ii) the variation of the cochlea's cross section, (iii) more correct treatment of boundary conditions and (iv) terms involving the derivative of the BM width.

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FIGURE 2. Cartoon of the cochlea


FIGURE 3. The amplitude of basilar membrane displacement, $\operatorname{real}(\tilde{w})$ given by equation 31 at $y=0$, for different transversal wave number $n$ and for some values of dimensionless frequencies $\frac{\omega}{\omega_{0}}$ : (a) $n=1$, (b) $n=2$ and (c) $n=3$. For cases (a) and (b) decreasing values of $\frac{\omega}{\omega_{0}}$ equals to $\{50,45,40,35,30,25,20,15,10,5,1\}$. For case (c), $\frac{\omega}{\omega_{0}}$ equals to $\{80,70,60,50,40,30,20,10,1\}$.


FIGURE 4. The amplitude of basilar membrane displacement, $\operatorname{real}(\tilde{w})$ given by equation 31 at $y=0$, for different transversal wave number $n$ and for some values of dimensionless frequencies $\frac{\omega}{\omega_{0}}$ : (a) $n=4$, (b) $n=4$ and (c) $n=6$. Decreasing values of $\frac{\omega}{\omega_{0}}$ equals to : (a) $\{80,70,60,50,40,30,20,10,1\}$, (b) $\{60,50,40,30,20,10,1\}$, (c) $\{40,30,20,10,1\}$.


FIGURE 5. (a) Real part versus imaginary part of the instable temporal eigenvalues of the system. It was obtained by the asymptotic analysis (local analysis) for some values of the wave number $k$ distributed with a constant step in the range $4 \leq k \leq 12$. The cross section, where the local analysis has been done, is located at different values of $x$. Increasing values of $x$ equal to $\{1.67,3.33,5,6.67,8.33,10,11.67,13.33,15\}$. (b) Group velocity related to the eigenvectors of the instable modes versus the wave number, in some cochlea locations, in the same cross section. Increasing values of $x$ (from the top to bottom) equal to $\{1.67,3.33,5,6.67,8.33,10,11.67,13.33,15\}$.


FIGURE 6. Maximum and minimum of the amplitude of basilar membrane displacement, $\operatorname{real}(\tilde{w})$ given by equation 31 , versus real part of the dimensionless frequency, that is, $\operatorname{Re}\left(\frac{\omega}{\omega_{0}}\right)$ for some transversal wave numbers. The set of envelopes in this figure are obtained with $\frac{\omega}{\omega_{0}}=\operatorname{Re}\left(\frac{\omega}{\omega_{0}}\right)\left(1+\frac{1}{2} i\right)$. The numerical simulation suggests that the solution does not exist outside the envelope nor for $n \geq 7$.


FIGURE 7. The amplitude of basilar membrane displacement, $\operatorname{real}(\tilde{w})$ given by equation 31 , for three complex frequencies: (a) $\frac{\omega}{\omega_{0}}=$ $35\left(1+\frac{1}{2} i\right)$, (b) $\frac{\omega}{\omega_{0}}=15\left(1+\frac{1}{2} i\right)$, (c) $\frac{\omega}{\omega_{0}}=5\left(1+\frac{1}{2} i\right)$. The BM's displacement is obtained with transversal wave number $n=1$. Only a portion of the BM is set in motion: near the base when the frequency is high, in the middle when the frequency is moderate and near the apex when the frequency is low.


FIGURE 8. The amplitude of basilar membrane displacement, $\operatorname{real}(\tilde{w})$ given by equation 31 , versus the cochlea's axial distance, for three wave numbers $n$ and for two complex frequencies: (a) $n=2$ and $\frac{\omega}{\omega_{0}} \in\left\{50\left(1+\frac{1}{2} i\right), 45\left(1+\frac{1}{2} i\right)\right\}$, (b) $n=$ 3 and $\frac{\omega}{\omega_{0}} \in\left\{40\left(1+\frac{1}{2} i\right), 35\left(1+\frac{1}{2} i\right)\right\}, \quad$ (c) $n=4$ and $\frac{\omega}{\omega_{0}} \in$ $\left\{35\left(1+\frac{1}{2} i\right), 30\left(1+\frac{1}{2} i\right)\right\}$.


FIGURE 9. The amplitude of basilar membrane displacement, $\operatorname{real}(\tilde{w})$ given by equation 31 , for two wave numbers $n$ and for two complex frequencies: (a) $n=5$ and $\frac{\omega}{\omega_{0}} \in\left\{25\left(1+\frac{1}{2} i\right), 20\left(1+\frac{1}{2} i\right)\right\}$, (b) $n=6$ and $\frac{\omega}{\omega_{0}} \in\left\{20\left(1+\frac{1}{2} i\right), \quad 15\left(1+\frac{1}{2} i\right)\right\}$,


[^0]:    ${ }^{1}$ Wentzel, Kramers, and Brillouin method

[^1]:    ${ }^{2}$ This assumption does not induces any error as far as the equation of motion are satisfied which is the case here. However, it leads to a particular solution of the equation of motion. Important discoveries have been found by this kind of assumption in fluid mechanics as, for instance, the shocked flow in the De Laval nozzle

[^2]:    ${ }^{3}$ The derivative of the function $r(x)$ appears via equation 29 when uper $W$ case is eleminated by from equation 26 by equation 29 and has to be kept when large variation of the function $r(x)$ is considered

