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NATURAL CONVECTION WITH POROUS MEDIUM IN A NARROW HORIZONTAL CYLINDRICAL ANNULUS WITH CONSTANT HEAT GENERATION

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ABSTRACT

We consider the two-dimensional problem of steady natural convection in a narrow horizontal cylindrical annulus filled with porous medium due to a constant temperature variation on the outer and adiabatic conditions at the inner boundaries with constant volumetric heat flux. The solution is expanded in powers of a single combined similarity parameter, which is the product of the gap ratio to the power of two, and Rayleigh number. The series is extended by means of symbolic calculation up to 28 terms. Analysis of these expansions allows the exact computation for arbitrarily accuracy up to 50000 figures. Although the range of the radius of convergence is small, but Pade approximation leads our results to be good even for much higher value of the similarity parameter.

Keywords: Micro gap, Nonlinear Equation, Symbolic Calculation

NOMENCLATURE

g	Gravity
$K = \varepsilon^2 Ra$	Similarity parameter
R	Outer radius
$Ra = \frac{g\beta\Delta T_a(KK)R_i}{\nu\alpha}$	Rayleigh number
$Rq = \frac{g\beta\Delta T_q(KK)R_i}{\nu\alpha}$	Rayleigh number
R_i	Inner radius
T	Temperature
β	The coefficient of thermal expansion
ε	Ratio of gap width to inner radius
ν	Kinematics viscosity
k	Effective thermal conductivity
KK	The permeability of the media
α	Thermal diffusivity
ψ	Stream function
ΔT_q	Imposed temperature difference
ΔT_a	Imposed temperature difference
V	Velocity
Nu	Nusselt number
Nu1	Other Nusselt number

Superscript

'	Dimensional variable

Subscripts

r	Value at reference temperature T_r

INTRODUCTION

This is problem of the two-dimensional problem of steady natural convection in a narrow horizontal cylindrical annulus filled with porous medium due to a constant temperature variation on the outer and adiabatic conditions at the inner boundaries with constant volumetric heat flux. The solution is expanded in powers of a single combined similarity parameter which is the product of the Gap ratio to the power of two, and Rayleigh number ($K = \varepsilon^2$Ra) and the series extended by means of symbolic calculation up to 28 terms. Analysis of these expansions allows the exact computation for arbitrarily accuracy up to 50000 figures. Although the range of the radius of convergence is small but Pade approximation lead our result to be good even for much higher value of the similarity parameter we have found extending terms exactly by means of symbolic calculation up to 28th order. Then we tried to make analytic continuation by using Pade approximation as have been done by [1], [2], [3]. In other words, we have solved the nonlinear partial differential equation exactly by means of computer and that is a real success.

The natural convection in a narrow horizontal cylindrical annulus filled with porous medium has received much attention because of the theoretical interest of [4], [5], [6], [7] for isothermal surfaces and [8] for isothermal inner and sinusoidal outer boundaries. As far as the numerical works for the case of isothermal surfaces a parameter study of diameter-ratio effects on the heat transfer coefficient was performed by [4] and angle of heating by [8] and other related problem of natural convection non-Darcy effect by [9] and finally experimental work of [4]. The question of hydrodynamic instability induces steady or oscillatory flows have been subject of many studies for example [4] and [10]. I hope the present exact solution of steady flow will help to answer such question much more clearly. We recently have done the same present approach of symbolic calculation for laminar flow through heated horizontal pipe [3] and similar work done for concentric spheres [11]. There are lots of wide engineering application such as thermal energy storage systems, cooling of electronic components and transmission cables and this was subject of study by [12] for viscous medium. It makes the problem of narrow annulus under consideration very important.

Particularly gratifying is our ability to extract from the perturbation series for small gap order 10^{-6} of the inner radius (micro size) the behavior for finite gap, of course the gap even as large as the inner radius. There are lots of wide engineering application such as thermal energy storage systems, cooling of electronic components and transmission cables. It makes the problem under consideration very important. Finally in simple words for Nusselt number which is the rate of heat transfer we will give a simple formula for all values of Rayleigh number up to K=30.

1. STATEMENT OF PROBLEM

The governing equations for porous materials with Darcy's law as has been used in [13] can be written as:

$$\nabla \dot{V} = 0 \tag{1}$$

$$\dot{V} = -\frac{\mathrm{KK}}{\mu}(\nabla \dot{P} - \dot{\rho} g J) \tag{2}$$

$$q''' + \dot{\rho} c (V.\dot{\nabla})\dot{T} = k\nabla^2 \dot{T} \tag{3}$$

$$\rho^{\cdot} = \rho_r\ (1-\beta(\mathrm{T}-T_r)) \tag{4}$$

Where $\dot{V}$ is the velocity vector, $\dot{\rho}$ density,T temperature, μ viscosity, $\dot{P}$ pressure = (cos θ ,-sin θ) is a unit vector in the direction of gravity and Rq $= \frac{\mathrm{g}\beta\Delta T_q(\mathrm{KK})\mathrm{R_i}}{\nu\alpha}$ is defined the internal Rayleigh number. The equations (5),(6) have been non-dimensionalzed by scaling length, velocity and temperature using the inner radius of cavity as the length scale, $\frac{k}{\mathrm{Ri}(\rho c)_r}, \Delta T_q$. ΔT_q is calculated from uniform heat generation q''', $\Delta T_q \sim \frac{q'''\mathrm{R_i}^2}{\alpha}$ is somehow certain gradient of temperature across the cavity, β the coefficient of volumetric expansion of the fluid) and g the acceleration due to gravity. Introducing the stream function in order to satisfy Eq. (1), eliminating the pressure from Eq. (2) and writing the resulting equation in cylindrical polar coordinates leads to the dimensionless equations as has been used in [13]:

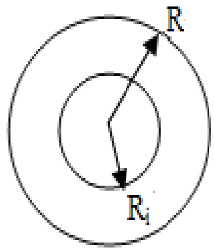

Fig. 1 Geometry of the problem

$$\nabla^2\psi = Rq\left(\frac{\partial T}{\partial r}\sin\theta + \cos\theta\ \frac{\partial T}{\partial \theta}\frac{1}{r}\right) \tag{5}$$

$$\frac{1}{r}\left(\frac{\partial\psi}{\partial\theta}\frac{\partial T}{\partial r} - \frac{\partial\psi}{\partial r}\frac{\partial T}{\partial\theta}\right) + 2 = \nabla^2 T \tag{6}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2},$$

$$T(1) = \frac{Ra}{Rq}$$
$$T(0) = 0$$
$$\Psi(1) = \Psi(0) = 0 \tag{7}$$

If we define (a) ratio of the outer and inner radius as shown in Figure 1. ($a = \frac{R}{R_i}$), then we change variables:

$$a = 1 + \varepsilon \qquad r = 1 + \varepsilon\eta \qquad 0 \le \eta \le 1$$

Using the above definition of Rayleigh number then Eq. (4) and Eq. (5) in the present notation becomes:

$$
\begin{cases}
\nabla^{*2}\psi - \left(\sin\theta \dfrac{\partial T}{\partial \eta} + \dfrac{\cos\theta}{1+\varepsilon\eta} \dfrac{\partial T}{\partial \theta} \right) = 0 \\
\nabla^{*2} T = \dfrac{\varepsilon^2 Ra}{1+\varepsilon\eta} \left(\dfrac{\partial \psi}{\partial \theta}\dfrac{\partial T}{\partial \eta} - \dfrac{\partial \psi}{\partial \eta}\dfrac{\partial T}{\partial \theta} \right) + 2
\end{cases} \tag{8}
$$

Where

$$
\nabla^{*2} = \frac{\partial^2}{\partial \eta^2} + \frac{1}{1+\varepsilon\eta}\left(\frac{\partial}{\partial \eta}\right) + \frac{1}{(1+\varepsilon\eta)^2}\left(\frac{\partial^2}{\partial \theta^2}\right)
$$

With boundary conditions at $\eta = 0{,}1$:

Let $Rq = Ra$

$$
\begin{aligned}
&T(1) = 1 \\
&T(0) = 0 \\
&\Psi(1) = \Psi(0) = 0
\end{aligned} \tag{9}
$$

As Mansur [1] has pointed out now by analogy with Dean's treatment of flow through a curved pipe, and Mansour [2] flow through a slowly rotating pipe, one can analyze the double limit for:

$$
\begin{cases} \varepsilon \to 0 \\ Rq \to \infty \end{cases} \qquad K = \varepsilon^2 \mathrm{Rq} \text{ Fixed}
$$

We follow narrow gap approximation and set $K = \varepsilon^2 R_a$, some terms will disappear from Eq.(8). Thus we seek to solve the simplified equations:

$$
\begin{cases}
\dfrac{\partial^2 \psi}{\partial^2 \eta} = \left(\sin(\theta) \dfrac{\partial T}{\partial \eta} \right) \\
\dfrac{\partial^2 T}{\partial^2 \eta} = K \left(\dfrac{\partial \psi}{\partial \theta}\dfrac{\partial T}{\partial \eta} - \dfrac{\partial \psi}{\partial \eta}\dfrac{\partial T}{\partial \theta} \right) + 1
\end{cases} \tag{10}
$$

2. SERIES DERIVATION AND COMPUTER EXTENSION

We consider an approximation for narrow gap by expanding in powers of the similarity parameter *K*; however we observe that beyond two terms extension is difficult.

In this work we delegated the mounting algebra to the computer and, for our boundary condition (9) one can systematically improve on this approximation.

$$
\begin{cases}
T = T_0 + KT_1 + K^2 T_2 + K^3 T_3 + \ldots. \\
\psi = K\psi_1 + K^2\psi_2 + K^3\psi_3 + \ldots\ldots
\end{cases} \tag{11}
$$

We substitute the expansions (11) into our simplified Eq. (10) and then equating like powers of K gives for T_0 the equation:

$$
\frac{\partial^2 T_0}{\partial \eta^2} = 1
$$

subject to boundary condition (9) we have :

$$
\frac{\partial^2 T_0}{\partial \eta^2} = 2 \quad , \quad T_0 = \eta^2
$$

And similarly we obtain:

$$
\Psi_1 = (\frac{1}{4}) r(-1 + r^2) * Sin(\theta)
$$

$$
\Psi_2 = \frac{1}{2520}(r(-11 + 21r - 14r^4 + 4r^6) * Sin(2\theta))
$$

$$
\Psi_3 = (\frac{1}{59875200})(r((-1103 + 11649 * r - 27720r^3 + 8712r^4 + 5544r^5 + 9240r^6 - 7810r^8 + 1488 * r^{10})Sin(\theta) + 33(-3 + 53r - 264r^4 \quad 168r^5 + 120r^6 - 90r^8 + 16r^{10})Sin(3\theta)))
$$

$$
\Psi_4 = (\frac{1}{653837184000}) (r(2(439906 - 471571r - 2537535r^3 + 1440166r^4 + 3335332r^5 + 943800r^6 - 3513510r^7 - 1798940r^8 + 1009008r^9 + 1867320r^{10} - 825720r^{12} + 111744r^{14}) Sin(2\theta) + (-14504 + 239199r - 725725r^3 - 54054r^4 + 212212r^5 + 2076360r^6 - 1621620r^7 - 998140r^8 + 528528r^9 + 578760r^{10} - 253400r^{12} + 32384r^{14}) \quad Sin(4\theta)))
$$

$$
T_1 = (\frac{1}{90}) r(3 - 5r^3 + 2r^5)Cos(\theta)
$$

$$
T_2 = (\frac{1}{453600})(r(7(29 - 120r^2 + 72r^4 + 100r^5 - 105r^7 + 24r^9) + (53 - 660r^3 + 504r^4 + 420r^5 - 405r^7 + 88r^9) * Cos(2\theta)))
$$

$$T_3 = \left(\frac{1}{163459296000}\right)(r((-703943 - 11601590r^2 + 7065695r^3 + 20648628r^4 + 13873860r^5 - 34594560r^6 - 20682090r^7 + 12732720r^8 + 23723700r^9 - 12381460r^{11} + 1919040r^{13})Cos(\theta) + (239199 - 1451450r^2 - 135135r^3 + 636636r^4 + 7267260r^5 - 6486480r^6 - 4491630r^7 + 2642640r^8 + 3183180r^9 - 1647100r^{11} + 242880r^{13})Cos(3\theta)))$$

$$T_4 = \frac{1}{2000741783040000}(r(17(-22243673 - 3895600r^2 + 124053930r^3 + 261543528r^4 - 158046200r^5 - 899269800r^6 - 5674305r^7 + 1069971760r^8 + 508107600r^9 - 767763360r^{10} - 551651100r^{11} + 161199360r^{12} + 408400200r^{13} - 141344820r^{15} + 16612480r^{17}) + 4(-14748187 - 416222220r^2 + 501558990r^3 + 1979320812r^4 - 1466556000r^5 - 2893424820r^6 - 873777645r^7 + 4154190040r^8 + 3051391200r^9 - 3480750000r^{10} - 2493463440r^{11} + 771805440r^{12} + 1700635800r^{13} - 588294690r^{15} + 68334720r^{17})Cos(2\theta) + (-107027707 + 144276960r^2 - 110759250r^3 + 97593192r^4 + 1313615160r^5 - 2173022280r^6 - 699362235r^7 + 1583650640r^8 + 1635382320r^9 - 1416804480r^{10} - 1006314660r^{11} + 346832640r^{12} + 561418200r^{13} - 190690020r^{15} + 21211520r^{17})Cos(4\theta)))$$

It is possible to introduce an overall Nusselt number defined as:

$$Nu = \frac{1}{Nu^s}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{\partial T}{\partial \eta}\right)_{\eta=1} d\theta \quad ,$$

$$Nu1 = \frac{1}{Nu1^s}\int_0^{\pi} \left(\frac{\partial T}{\partial r}\right)_{\eta=1} d\theta,$$

$$Nu = (2\pi + \frac{29\pi}{64800}K^2 - \frac{22243673\pi}{117690693120000}K^4 - \frac{805368812413961\pi}{6904174470373955174400000}K^6 + \ldots$$

$$Nu1 = 2\pi - \frac{1}{9}K^1 + \frac{29\pi}{64800}K^2 + \frac{33227}{1571724000}K^3 + \frac{22243673\pi}{117690693120000}K^4 - \frac{141462161321}{12665272623696000000}K^5 + \frac{805368812413961\pi}{6904174470373955174400000}K^6 + \frac{267696499615770139}{284208772942382416128000000000}K^7 + \ldots$$

Where

$$Nu^s = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{\partial T_0}{\partial \eta}\right)_{\eta=1} d\theta \quad , \qquad Nu1^s = \int_0^{\pi} \left(\frac{\partial T_0}{\partial r}\right)_{\eta=1} d\theta$$

We have extended this expansion exactly by means of symbolic computation up to order of 28 we quote only the result up to order of 8 just because of numbers becomes too long. Of course for lack of space we omit showing the calculation further than this order if any reader interested to have more calculation please contacts the author. We mention that the author in [8] has calculated Nusselt number with another definition which is in the form of the absolute value we omit to show our calculation for this case.

3. ANALYSIS OF SERIES AND DISCUSSION

Pade approximants has been used in original forms to enable us to increase the range of applicability of the series as has been used and related refrences is cited in the works of Mansour [2] and Mansour [3]. This method does not necessarily require any information about the radius of convergence. The Pade approximants provide an approximation that is invariant under an Euler transformation of the independent variables. The theory of Pade approximants has been used extensively in Mansour [1]. Briefly stated, the Pade approximant is the ratio $P(K)/Q(K)$ of polynomials P and Q of degree m and n, respectively, that, when expanded, agrees with the given series through terms of degree m+n, and normalized by P(Q)=1. Such rational fractions are known to have remarkable properties of analytic continuation. The coefficients of the power series must be known to degree m+n. By equating like power of g(x) and $P(K)/Q(K)$, the linear system of m+n+1 equation must be solved to obtain the coefficients in the functional form $P(K)/Q(K)$.Pade approximation of orders [1/2], [2/3] for Nusselt number series are respectively:

pade[1/2]:

$$\frac{2\pi}{1-\frac{29*K^2}{129600}}$$

pade[2/3]:

$$\frac{2\pi+\pi\left(\frac{11343149*K^2}{8778369600}\right)}{1+\left(\frac{22243673*K^2}{52670217600}\right)}$$

pade[3/4]:

$$\frac{\left(2\pi+\frac{1544061207911017\pi*K^2}{998147062396804320}\right)}{1+\left(\frac{2743398976200871*K^2}{4990735311984021600}-\frac{73742696304558623*K^4}{2587197185732516797440000}\right)}$$

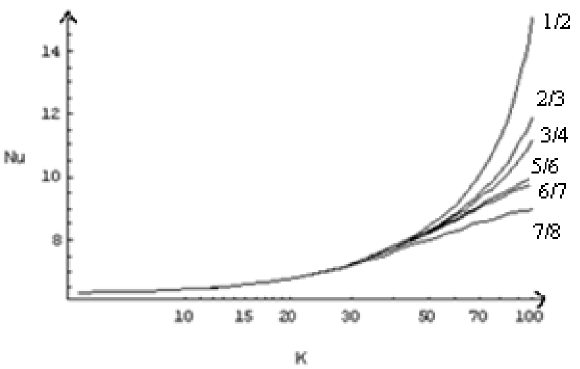

Fig. 2 Plots of [7/8],[6/7],[5/6],[3/4]and [2/3] of the Pade approximants for Nusselt number series versus *K*

When we form the ratios [7/8] and [6/7] and… Of the Pade approximants, it can be shown, they agreed up to the value $K \cong 30$. This conclusion is confirmed as is plotted in Fig 2.

4. CONCLUSION

This is problem of the two-dimensional problem of steady natural convection in a narrow horizontal cylindrical annulus filled with porous medium due to a constant temperature variation on the outer and adiabatic conditions at the inner boundaries with constant volumetric heat flux. The solution is expanded in powers of a single combined similarity parameter which is the product of the gap ratio to the power of two, and Rayleigh number ($K = \varepsilon^2 Ra$) and the series extended by means of symbolic calculation up to 28 terms. Analysis of these expansions allows the exact computation for arbitrarily accuracy up to 50000 figures. Although the range of the radius of convergence is small but Pade approximation lead our result to be good even for much higher value of the similarity parameter we have found extending terms exactly by means of symbolic calculation up to 28th order. Then we tried to make analytic continuation by using Pade approximation. In other words, we have solved the nonlinear partial differential equation exactly by means of computer and that is a real success.

As has been mentioned in the introduction we repeat again the natural convection in a narrow horizontal cylindrical annulus filled with porous medium has received much attention because of the theoretical interest of [4], [5], [6], [7] for isothermal surfaces and [8] for isothermal inner and sinusoidal outer boundaries. As far as the numerical works for the case of isothermal surfaces a parameter study of diameter-ratio effects on the heat transfer coefficient was performed by [4] and angle of heating by [8] and other related problem of natural convection non-Darcy effect by [9] and finally experimental work of [4]. The question of hydrodynamic instability induces steady or oscillatory flows have been subject of many studies for example [4] and [10]. I hope the present exact solution of steady flow will help to answer such question much more clearly. We recently have done the same present approach of symbolic calculation for laminar flow through heated horizontal pipe [3] and similar work done for concentrically spheres [11]. There are lots of wide engineering application such as thermal energy storage systems, cooling of electronic components and transmission cables and this was subject of study by [12] for viscous medium. It makes the problem of narrow annulus under consideration very important.

In general the heat transfer can be measured in steady natural convection in a narrow horizontal cylindrical annulus filled with porous medium, experimenters followed by theoreticians plot the Nusselt number versus of some dimensionless parameters, including Ra, ε, but in this work we combine Ra $,\varepsilon$ as $K = \varepsilon^2 Ra$.**.** It is worth to mention in the past that we were indeed able to find surprisingly the Pade approximation agrees very well qualitatively with result given in experiment refer for example to Mansour[3], but for this particular work and with these boundary conditions we have not found any experimental data to compare with. As the author in [1] has mentioned calculation in this type of problems suffers cancellation that it means, if one uses FORTRAN we cannot be able to get more than order up to 4 so it is very essential to use symbolic language like Maple as is used in this work. We repeat our series has small radius of convergence but by using Pade approximant we are able to go as far as K=30 and that is real success.

It is particularly gratifying is our ability to extract from the perturbation series for small gap (micro size) the behavior for finite gap, of course the gap even as large as the inner radius in refrence [13] the same problem for finite gap has been done and is consistence to our results ,we cannot compare our results with them because their boundary conditions is different from ours . Finally in simple words for Nusselt number which is the rate of heat transfer we gave a simple formula for all values of modified Rayleigh number up to $K \cong 30$ and this is in my view good achievement. Of course we found the Nusselt number which is the rate of heat transfer through a micro gap from macroscopic not microscopic view point. It would be interesting to have a work from microscopic investigation in order to compare with but we do not know anyone.

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