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Symbolic Calculation of Laminar Convection In Uniformly Heated Horizontal Pipe at High Prandtl Number

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ABSTRACT

We consider fully developed steady laminar flow through a uniformly heated horizontal pipe is simplified by assuming infinite Prandtl number. The solution is expanded in powers of a single combined similarity parameter which is the product of the Prandtl, Rayleigh, and Reynolds numbers and the series extended by means of symbolic calculation up to 16 terms. Analysis of these expansions allows the exact computation for arbitrarily accuracy up to 50000 figures. Although the range of exactness is almost the same order of the radius of convergence but Pade approximation lead our result to be good even for much higher value of the similarity parameter.

Keywords: Heat pipe, Nonlinear Equation, Symbolic Calculation

NOMENCLATURE

a	Radius
β	The coefficient of thermal expansion
g	Gravity
γ	the pressure gradient
ν	Kinematics viscosity
ψ	Stream function
T	Temperature
τ	Temperature gradient
k	Thermal diffusivity
$Re = \frac{\gamma a^3}{4\rho \nu^2}$	Reynolds number
$Pr = \frac{\upsilon}{K}$	Prandtl number
$Ra = \frac{\beta g \tau a^4}{k\upsilon}$	Rayleigh number
$\alpha = \frac{K}{\rho c_p}$	Thermal diffusivity

INTRODUCTION

We consider fully developed steady laminar flow through a uniformly heated horizontal pipe is simplified by assuming infinite Prandtl number. The solution is expanded in powers of a single combined similarity parameter which is the product of the Prandtl, Rayleigh, and Reynolds numbers. We hope this way of approach to this problem shed a light to obtain the boundary layer solution more effectively, also to use this exact results for a base in hydrodynamic instability.

1. STATEMENT OF PROBLEM

We consider Morton [1] problem of fully developed steady laminar flow through a horizontal pipe of radius **a,** wall is heated uniformly. We neglect dissipation, and the pressure term in the energy equation, and variation of density except in the buoyancy force, and take the kinematics viscosity ν and thermal diffusivity k as constants. We introduced cylindrical coordinates (**r**, θ ,**x**) with x increasing in the stream wise direction and θ measured from upward vertical. We mainly follow Morton's notation. A constant temperature gradient τ is maintained in the axial direction; and in fully developed flow the pressure has a constant (negative) gradient $\tilde{\gamma}$, Then the velocity component U, V, W in the fluid and the temperature decrement τ x **- T** are functions of **r** and θ only. The continuity equation can be satisfied by introducing a Stokes stream function ψ for the cross flow the Prandtl number **Pr** ,Raleigh number Ra ,and Reynolds number Re (Morton's $\boldsymbol{\sigma}$,**A**, and **B**) are defined as

$$Ra = \frac{\beta g \tau a^4}{k v} \quad ,Re = \frac{\gamma a^3}{4\rho v^2} \ , Pr = \frac{v}{K}$$

Here, β is the coefficient of thermal expansion and **g** the acceleration due to gravity Re is what woods & Morries [13] aptly call the pseudo Reynolds number based on the radius and maximum speed that would be produced by the pressure gradient in the absence of heating.

.

$$\begin{cases} \nabla^4\psi = Ra(\frac{\partial T}{\partial r}\sin\theta + \frac{\partial T}{\partial \theta}\frac{1}{r}\cos\theta) + \\ \qquad \frac{1}{Pr}\left[\frac{1}{r}\left(-\frac{\partial\psi}{\partial\theta}\frac{\partial}{\partial r} + \frac{\partial\psi}{\partial r}\frac{\partial}{\partial\theta}\right)\nabla^2\psi\right] \\ \nabla^2 T - Pr\frac{1}{r}\left(\frac{\partial\psi}{\partial\theta}\frac{\partial T}{\partial r} - \frac{\partial\psi}{\partial r}\frac{\partial T}{\partial\theta}\right) = -W \\ \nabla^2 W = \frac{1}{r}\left(\frac{\partial\psi}{\partial\theta}\frac{\partial W}{\partial r} - \frac{\partial\psi}{\partial r}\frac{\partial W}{\partial\theta}\right) - 4Re \end{cases} \tag{1}$$

Following Vandyke [2] he modified Morton's variables in order to clarify the dependence of the problem on the parameters Pr, Ra and Re .We refer ψ to k rather than ν, multiply Morton,s dimensionless temperature difference and axial velocity w by the Reynolds number **Re,** and then subtract the parabolic poiseuille velocity profile from w and divide the reminder by Pr. Thus we express the original dimensional variables in terms of new (barred) variables as:

$$\psi = k\bar{\psi}(r,\vartheta), \ddot{r} = a\bar{r}$$

$$W = \frac{v}{a}Re\left[(1-r^2) + \frac{1}{Pr}\bar{w}(r,\vartheta)\right]$$

$$T = \tau x - \tau a Pr Re \theta(r,\vartheta)$$

Then vortices equation, axial- momentum equation, and energy equation reduced, with the over bars dropped to:

$$\begin{cases} \nabla^4\psi = K(\frac{\partial T}{\partial r}\sin\theta + \frac{\partial T}{\partial \theta}\frac{1}{r}\cos\theta) + \\ \qquad \frac{1}{Pr}\left[\frac{1}{r}\left(-\frac{\partial\psi}{\partial\theta}\frac{\partial}{\partial r} + \frac{\partial\psi}{\partial r}\frac{\partial}{\partial\theta}\right)\nabla^2\psi\right] \\ \nabla^2 T = \frac{1}{r}\left(\frac{\partial\psi}{\partial\theta}\frac{\partial T}{\partial r} - \frac{\partial\psi}{\partial r}\frac{\partial T}{\partial\theta}\right) - (1-r^2) - \frac{1}{Pr}W \\ \nabla^2 W = \frac{1}{Pr}\frac{1}{r}\left(\frac{\partial\psi}{\partial\theta}\frac{\partial W}{\partial r} - \frac{\partial\psi}{\partial r}\frac{\partial W}{\partial\theta}\right) - 2\frac{\partial\psi}{\partial\theta} \end{cases} \tag{2}$$

Where

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\left(\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\left(\frac{\partial^2}{\partial\theta^2}\right)$$

$$K = \frac{\beta g \tau(-\gamma)a^7}{4\rho v^2 k^2} = (Pr)(Ra)(Re)$$

.

The boundary conditions on ψ and W require that the three components of velocity vanish at the wall and we use, Anderson' [3] version for temperature:

$$\frac{\partial T}{\partial r} = 0 \tag{3}$$

$$\frac{\partial\psi}{\partial r} = \frac{\partial\psi}{\partial\theta} = W = 0 \qquad at\ r = 1$$

The other boundary conditions on temperature are found in literature, Morton [1] uses:

(T=0 at r=1) (4)

2-SERIES DERIVATION AND COMPUTER EXTENSION

Morton [1] sought an approximation for slight heating by expanding in powers of the Raleigh number Ra, with the Prandtl and Reynolds numbers as parameters. However he observed that Ra and Re appeared only as a product, so he effectively expanded in powers of RaRe, with coefficients that are polynomials in Pr of continually increasing order. Hence he computed (to second order) a double power series in RaRe and Pr.

Anderson[3] delegated the mounting algebra to the computer and, for his boundary condition of strictly uniform heat transfer calculated seven terms of the single series in power of Vandyke' K .We here extend the computation for Morton [1]' boundary conditions(4).

We have scaled K by 4608.$\bar{K} = \frac{K}{4608}$ we omit the barred.

$$T = T_0 + T_1 K + T_2 K^2 + \cdots = \sum_{n=0}^{\infty} T_n K^n. \tag{5}$$

$$\psi = \psi_1 K + \psi_2 K^2 + \cdots = \sum_{n=1}^{\infty} \psi_n K^n. \tag{6}$$

$$W = W_1 K + W_2 K^2 + \cdots = \sum_{n=1}^{\infty} W_n K^n. \qquad (7)$$

We substitute the expansions (5) into our simplified equation (2), and then equating like powers of K gives for T_0 the equation:

$$\nabla^2 T_0 = -(1 - r^2)$$

,

$$T_0 = (\frac{1}{16}) * (3 - 4 * r^2 + r^4)$$

And similarly we obtain:

$$T_1 = (\frac{1}{960}) r(-265 + 600r^2 - 520r^4 + 225r^6 - 42r^8 + 2r^{10}) Cos(\theta)$$

$$T_2 = (\frac{1}{67737600})((-1 + r^2)(70(-1 + r^2)^3(-129578 + 149488r^2 - 81770r^4 + 20000r^6 - 1855r^8 + 56r^{10}) + 9r^2(-843810 + 1796422r^2 - 1789734r^4 + 989826r^6 - 318964r^8 + 57020r^{10} - 4377r^{12} + 119r^{14} * Cos(2\theta)))$$

$$T_3 = (\frac{1}{42620497920000})(r * (-1 + r^2) * ((-28908036004343 + 89303607885997 * r^2 - 159218017782033 * r^4 + 182780479532727 * r^6 - 142114858068393 * r^8 + 76330702368711 * r^{10} - 28276773873129 * r^{12} + 7059555397971 * r^{14} - 1121837546754 * r^{16} + 103362495026 * r^{18} - 4927527100 * r^{20} + 93439500 * r^{22}) * Cos(\theta) + 2 * r^2 * (663433868228 - 2232649979572 * r^2 + 3385409102492 * r^4 - 2918866549468 * r^6 + 1577601219182r^8 - 565976306479r^{10} + 136967349221r^{12} - 21587511529r^{14} + 1954717181r^{16} - 92135484r^{18} + 1708200r^{20}) Cos(3\theta)))$$

$$T_4 = (\frac{1}{2215379385483264000000})((-1 + r^2) * (119 * (-1 + r^2)^3 * (65523439358768 6285 - 1581723242432392660r^2 + 2403119797359946430r^4 - 2503882529582165500 * r^6 + 1841549049129315815 * r^8 - 964385493544330432 r^{10} + 358531626747153992 * r^{12} - 93203614663942600 * r^{14} + 16336343698162985 * r^{16} - 1826985582656100 * r^{18} + 122484834180150 * r^{20} - 4446928876260 * r^{22} + 66972680775r^{24}) + 2r^2(6402958491154787260 3 - 2373109513338981022 97 * r^2 + 4865026223739184363 63 * r^4 - 6665315217899203186 37 * r^6 + 6509627261539977199 03 * r^8 - 4657667520999964126 25 * r^{10} + 2474475300114720146 65 * r^{12} - 980112599121103432 15 * r^{14} + 287626566721909200 62 * r^{16} - 61381497420802973 58r^{18} + 918928830067977312 * r^{20} - 91338059671941408 * r^{22} + 5623485554098632 * r^{24} - 191858090450520 * r^{26} + 2753336483520 * r^{28}) * Cos(2\theta) + r^4 * (-2301324357855328399 + 984796733160577960 1 * r^2 - 1864963422085822435 9 * r^4 + 2063334134622445372 1 * r^6 - 1511156650966383733 9 * r^8 + 787912281165477755 7 * r^{10} - 302111735578758417 9 * r^{12} + 854763064512335853 * r^{14} - 176270215073305479 * r^{16} + 25806371434090121 * r^{18} - 2522427112398319 * r^{20} + 154057230838385 * r^{22} - 5179857931192 * r^{24} + 72347311400 * r^{26}) * Cos(4\theta)))$$

$$W_1 = (-(\frac{1}{40})) * r(49 - 100 * r^2 + 70 * r^4 - 20 * r^6 + r^8) Cos(\theta)$$

$$W_2 = -(\left(\frac{1}{705600}\right) * (r^2 * (-362375 + 883512 * r^2 - 807975 * r^4 + 352800 * r^6 - 76440 * r^8 + 11340 * r^{10} - 882 * r^{12} + 20 * r^{14}) * Cos(2\theta)))$$

$$W_3 = -((\frac{1}{2663781120000}) * (r * (-1 + r^2) * (13 * (178444880654 - 238588598791 * r^2 + 163442951129 * r^4 - 73424894371 * r^6 + 28435003769 * r^8 - 8534899663 * r^{10} + 1887048137 * r^{12} - 286977013 * r^{14} + 26164937 * r^{16} - 1180458 * r^{18} + 19950 * r^{20}) Cos(\theta) + 99r^2(-2871848037 + 5256846563 * r^2 - 3988178785 * r^4 + 1675118855 * r^6 - 522682465 * r^8 + 116119985 * r^{10} - 17563565 * r^{12} + 1638475 * r^{14} - 71870 * r^{16} + 1190 * r^{18}) * Cos(3\theta)))) Cos(\theta)$$

$$\psi_1 = r * (-10 + r^2) * (-1 + r^2)^2 * Sin(\theta)$$

$$\psi_2 = (1/2800) * (r^2 * (-1 + r^2)^2 * (10518 - 4614 * r^2 + 1254 * r^4 - 158 * r^6 + 5 * r^8) Sin(2\theta))$$

$$\psi_3 = (\frac{1}{310464000}) * (r * (-1 + r^2)^2 * ((2527475633 - 2254713278 * r^2 + 1576474011 * r^4 - 765665860 * r^6 + 253094581 * r^8 - 54574698 * r^{10} + 6681503 * r^{12} - 384176 * r^{14} + 7980 * r^{16}) * Sin(\theta) + 2 * r^2 * (-125056840 + 105464218 * r^2 - 56089176 * r^4 + 19043726 * r^6 - 4100672 * r^8 + 519975r^{10} - 29158r^{12} + 595r^{14}) Sin(3\theta)))$$

$$\psi_4 = (\frac{1}{1616027212800000})(r^2 * (-1 + r^2)^2 * (10 * (-1388767633985622 + 1345082043007442 * r^2 - 1118580630983720 * r^4 + 730015378063424 * r^6 - 363160479631140 * r^8 + 134772718029544 * r^{10} - 36324299035645 * r^{12} + 6843101163778 * r^{14} - 841451107404 * r^{16} + 61093920194 * r^{18} - 2328014168 * r^{20} + 35781480 * r^{22}) * Sin(2\theta) + 13 * r^2 * (14886639996515 - 22955915389310 * r^2 + 20291876963265 * r^4 - 11367785111680 * r^6 + 4254620203075 * r^8 - 1106490735870 * r^{10} + 201306182797 * r^{12} - 24592147336 * r^{14} + 1755891906 * r^{16} - 66257052 * r^{18} + 996450 * r^{20}) * Sin(4\theta)))$$

In Andersons [3] version of the problem (boundary condion(3) , Nusselt number is given by:

$$\frac{-360715 + 114766 * p + 360715 * p^2}{1344000}$$

This agrees with the result given by Anderson [3]. We have extended this expansion exactly by means of symbolic computation up to order of 16 we quote only the result up to order of 10 just because of numbers becomes too long.

$$Nu = 1 - \frac{45187 * K^2}{135520} + \frac{52042924793159 * K^4}{85384499200000} - \frac{56631973555239665523718717691K^6}{29818120376850540134400000000} + \frac{(\frac{17327867297484400047436}{36530363281559346663919}) * K^8}{(\frac{199157533173154372756 89}{498299596800000000000})} - \frac{(\frac{14123865201456682158634520597 1}{66675052083531824886287722969 3})K^{10}}{(\frac{20250408686093245545741502823 5}{16510512742400000000000000000})} + \cdots$$

Of course this agrees with the result given numerically by great work of Vandyke [2].

We have extended this expansion exactly by means of symbolic computation up to order of 28 we quote only the result up to order of 8 just because of numbers becomes too long. Of course for lack of space we omit showing the calculation further than this order if any reader interested to have more calculation please contacts the author. We mention that the author in [8] has calculated Nusselt number with another definition which is in the form of the absolute value we omit to show our calculation for this case.

3. ANALYSIS OF SERIES AND DISCUSSION

Pade approximants has been used in original forms to enable us to increase the range of applicability of the series as has been used in the works of Mansour [4] and Mansour[5]. This method does not necessarily require any information about the radius of convergence. The Pade approximants provide an approximation that is invariant under an Euler transformation of the independent variables. The theory of Pade approximants has been used extensively in Mansour [4]. Briefly stated, the Pade approximant is the ratio P(x)/Q(x) of polynomials P and Q of degree m and n, respectively, that, when expanded, agrees with the given series through terms of degree m+n, and normalized by P(Q)=1. Such rational fractions are known to have remarkable properties of analytic continuation. The coefficients of the power series must be known to degree m+n. By equating like power of g(x) and P(x)/Q(x), the linear system of m+n+1 equation must be solved to obtain the coefficients in the functional form P(x)/Q(x).

Pade approximation of orders [1/2],[2/3],[3/4] are respectively:

pade[1/2]:

$$\frac{1}{1+\frac{45187 * K^2}{135520}}$$

pade[2/3]:

$$\frac{\left(1\,1+\frac{56634072471522629 * K^2}{37893716981120000}\right)}{1\ +\left(\frac{572472172724749 * K^2}{313171214720000}\right)}$$

pade[3/4]:

$$\frac{\left(1\,1+\frac{60716678778498686407399616253121 * K^2}{1977784734008417227024664332800 0}\right)}{1\ +\left(\frac{556291645270773700219688293801 * K^2}{163453283802348531159063168000}+\frac{44956201037082368912294206323956 53 * K^4}{855841393989096909148854747648000 00}\right)}$$

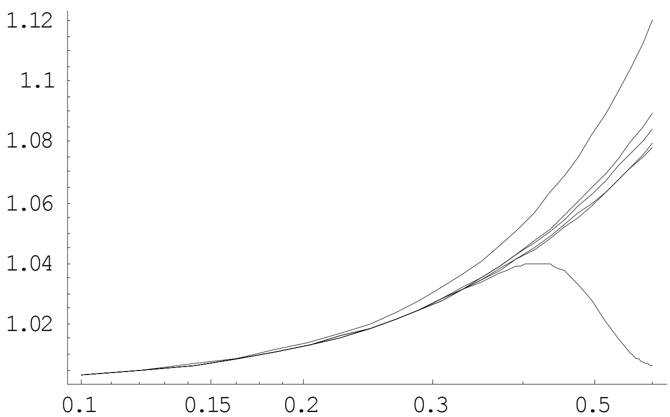

Fig. 2 Plots of [7/8],[6/7],[4/5],[3/4]and [2/3] of the Pade approximants for Nusselt number series versus *K/4608*

When we form the ratios [7/8] and [6/7] and… Of the Pade approximants, it can be shown, they agreed up to the value $K \cong 1000$. This conclusion is confirmed as is plotted in Fig 2.

4. CONCLUSION

In this problem, we have found 16 terms exactly by means of symbolic calculation. Then we tried to make analytic continuation by using Pade approximation. By doing so, we are successful to increase the range of the validity of the series from zero up to $K \cong 1000$ and these are exact calculations up to arbitrarily figure of accuracy. In other words, we have solved the nonlinear partial differential equation exactly by means of computer and that is a real success. We have discussed in detail only the boundary condition of zero conductivity and a Nusselt number referred to the bulk temperatures.

The heat transfer has been measured in horizontal heated pipes using water, air, and ethylene glycol, different experimenters followed by theoreticians have plotted the Nusselt number versus a variety of dimensionless parameters , including RaRe, Gr, and Gr Pr, where the Grashof number Gr is (aside from possible factors of 2 in the definitions) equal to Ra Re/Nu. Anderson [3] has criticized the last choice because both Gr and Nu contain the unknown average wall temperature. he emphasizes that the proper independent parameter is Gr Nu Pr ,which is equivalent to our K=Pr Ra Re. Anderson has converted to that parameter the measurements of Shannon & Depew[6] in water ,and the finit-difference calculations of Newell & Burgles[7] .

It is particularly gratifying is our ability to extract from the perturbation series for slight heating the behavior for finite heating, of course hopefully to infinity (boundary layer limit).

We just make two comments first that The correlation K=PrReRa was first observed by Anderson [3]. It has been independently rediscovered by Cheng. Hwang &Akiyana [12] and Woods & Morris [13] and second related works of Morcos et al[14] and Iqbal,& Stachiewicz[15] whom they used the other boundary condition(4) .

As trefethen [9] observed, the flow through a heated horizontal pipe is qualitatively similar in a number of respects to that through a coiled pipe or a rotating pipe, From a physical point of view Body forces produce in each case a double spiral secondary motion .transition to turbulence is thereby delayed to higher Reynolds number (for reasons that are by no means clear). of the boundary layer analyses of Mori & Futagami [10] and Siegwarthet al[11] . As Vandyke[2] has mentioned For the loosely coiled pipe the friction grows eventually as the 1/4 power of the practical Dean number based on the actual mean flow speed down the pipe, rather than the 1/2power predicted by four different approximate boundary layer models. For the rotating pipe Mansour [5], the friction grows as the 1/8 power, rather than the 1/4 power of his similarity variable to different boundary layer approximation. And for the heated pipe Vandyke[2] has found the heat transfer increasing as the 2/15 power, not the1/5 power of the boundary layer analyses of Mori & Futagami [10] and siegwarth et al.[11]. These controversial results made us to do exact calculation in this article and we hope first attempt for exact calculation of much higher Raleigh number.

Our approximation of high Prandtl number has proven not to be very restrictive. However, it would not be acceptable for a liquid metal like mercury, with Pr= 0.02.

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