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THE FLUID MECHANICS OF THE EYE AND THE ROLE OF THE MUCUS LAYER

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ABSTRACT

We discuss the role of the mucus layer in the eye. The tear film in the eye is composed of a mucus layer, an aqueous layer and a lipid layer. While the aqueous and lipid layers are Newtonian, the mucus layer is non-Newtonian. It is commonly believed that the mucus layer serves as a lubricant for the cornea. However, we hypothesize that it serves a greater purpose as a protective layer from foreign particles; the normal stress effects of a viscoelastic fluid under the blinking motion of the eyelid would act towards pushing out any particle embedded in this layer. To prove this hypothesis, we mathematically study the fluid mechanics of a viscoelastic, shear thinning fluid modeled by a generalized second grade fluid. As a first step, we investigate the flow and stresses induced by a shearing motion (part of a blink cycle) and its effect upon an embedded particle, which is modeled by the Wiberg-Smith equation.

INTRODUCTION

The human eye is composed of a composite sphere created by the combination of two spheres of radius 12mm and 9mm [19]. This sphere is covered by a film measuring about 10μm thick [5]. This tear film is contained by the skeleton and muscles of the face and skull, and the eyelids. The eyelid is a semi rigid structure that moves parallel to the surface of the eye ball. The movement of the upper eyelid is rapid and in the vertical direction and is caused by the direct effects of muscle contractions and relaxations. The lower eyelid moves in a much less dramatic, side to side motion and a more limited extent than the upper eyelid [4]. The blinking of the eyelids causes the obvious affects of spreading the tear film into a smooth layer, and permits tear film drainage [28].

The nature of the blink cycle and its effects on the precorneal tear film [14], and the bulk motion of the precorneal tear film is understood [16], though details are still up for discussion. The motion and stresses within the tear film itself, at the microscale level, has not been significantly recorded. We intend to do this by using models based on linear slider bearings and lubrication theory. The eyelid's relative dimensions, with the radius of the eyeball being about 1000 times the size of the separation between the eyelid and the eyeball, lead us to conclude that it is a reasonable assumption to consider the motion as being straight line motion between almost parallel plates.

The fluid that serves as the lubricating layer (see figure 1) consists of a very thin lipid layer, an aqueous layer, and a mucus layer [6]. The lipid layer serves mostly to reduce evaporation of the aqueous layer resided mostly between the two edges of the eyelid and is of no concern to us [1]. The aqueous layer is an enriched saline solution and behaves strictly Newtonian [12]. The mucus layer lies between the aqueous layer and the surface of the eyeball. Its molecular composition gives

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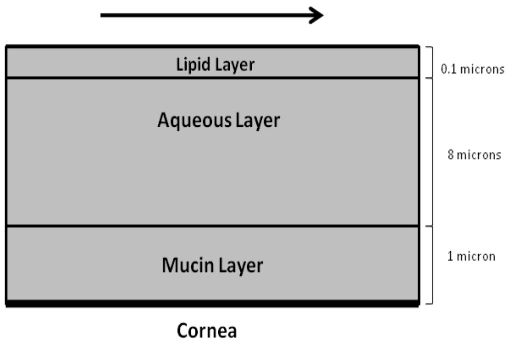

FIGURE 1. A SCHEMATIC OF THE TEAR FILM LAYERS IN THE HUMAN EYE. THE ARROW ON THE TOP INDICATES THE DIRECTION OF THE EYELID'S MOTION.

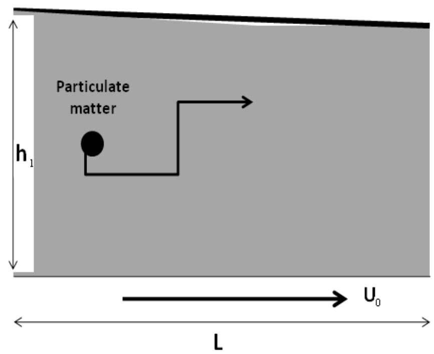

FIGURE 2. A SCHEMATIC OF THE MODEL GEOMETRY THAT WE USE IN OUR STUDY. IN THIS PAPER, THE UPPER PLATE IS HELD FIXED WHILE THE BOTTOM IS TAKEN TO BE MOVING IN KEEPING WITH THE CONVENTION IN PREVIOUS PAPERS.

it a shear dependent behavior, thus giving it a non-Newtonian character [20]. In addition to serving as a lubricant for the eyelid and as an adhesive that keeps the aqueous layer coating of the eyeball in place, it also serves in the role of a protector of the eye. Also, along with certain molecules and enzymes that work to chemically preventing disease from reaching the eye, there is some unclarified process whereby mucin moleculars wrap around unwelcome particles and serve to remove them from the tear film [15]. How these molecules are driven by the blink cycle is the subject of this paper.

THE MATHEMATICAL MODEL

In this study, we consider two aspects of the mechanics of the eye: (i) the first one regards the fluid mechanics of the tear film and (ii) the second concerns the induced motion of any tracer particle embedded in the fluid layers. As we discussed in the introduction, the tear film contains multiple fluid layers, each with different characteristic properties. The full problem as stated above is quite complex and we therefore consider a simplified first step in examining simply the mucus layer of the eye. Our hypothesis is that this layer in particular being non-Newtonian serves the purpose of transporting out any foreign material, away from the cornea. Figure 2 shows the geometry that we will assume in this current study. Specifically we will examine the flow and stresses produced in the fluid due to a steady shearing motion, which represents a portion of the blinking motion of the eye.

The Tear Film

Rheological studies [20, 26] indicate that the mucus layer has viscoelastic and shear-rate dependent viscosity. Most of the rheology performed on human tears is done for the combined occular fluid indicating a shear-thinning viscosity at high shear rates. There is evidence from mucus studies in general that at low shear rates, viscosity displays a shear thickening behavior which then becomes strongly shear thinning beyond a certain critical shear rate [15, 25]. Therefore we model our fluid as a shear thinning viscoelastic non-Newtonian fluid modeled by the generalized second grade fluid [17]. The model used here is an ad-hoc generalization of the more well known second grade fluid but nevertheless is the simplest incompressible, non-Newtonian model than can be constructed which can qualitatively capture the features of the mucus layer.

$$T = -p\mathrm{I} + \eta(\dot{\gamma})\mathrm{A}_1 + \alpha_1\mathrm{A}_1 + \alpha_2\mathrm{A}_2 = -pI + \tau \qquad (1)$$

where p is the indeterminate part of the stress tensor due to the constraint of incompressibility, η is the shear rate dependent viscosity and α_1 and α_2 are material moduli which are usually referred to as the normal stress coefficients. The kinematical tensors A_1 and A_2 are defined through

$$\mathrm{A}_1 = \mathrm{L} + \mathrm{L}^T \qquad (2)$$

$$\mathrm{A}_2 = \frac{d}{dt}\mathrm{A}_1 + \mathrm{A}_1\mathrm{L} + \mathrm{L}^T\mathrm{A}_1 \qquad (3)$$

$$\mathrm{L} = \text{grad } \mathrm{u} \qquad (4)$$

and

$$\eta(\dot{\gamma}) = \eta_0(1+k|\mathrm{A}_1|^2)^q \tag{5}$$

with η_0 representing the zero shear rate viscosity, k the consistency index and q the shear rate index which lends the fluid thickening ($q>0$), thinning ($q<0$) or constant viscosity ($q=0$) behavior. In this study we will consider analytical solutions to the problem which are derived for the cases of $q=\pm\epsilon$, where ϵ is very small [1]. Using the Binomial expansion, upto first order in q we have

$$\eta(\dot{\gamma}) = \eta_0(1+kq|\mathrm{A}_1|^2)+O(q^2). \tag{6}$$

Following the approach outlined by Bujurke et al. [2] and Huang et al. [13] and others [4,7–9,11] we begin by considering a velocity field given by $\mathrm{w} =< u(x,y), v(x,y), 0 >$ with the incompressibility condition

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0. \tag{7}$$

Henceforth, we will consider non-dimensional quantities (see figure 2) with characteristic length taken to be L, height is h_1 and characteristic velocity U_0 is the velocity of the moving plate. Also the Deborah number is given by $De=\frac{\alpha_2 U}{\eta_0 L}$. Using the fact that the flow profiles for second order fluid are similar to those of Newtonian fluids [13, 24], let us assume that the x-component of the velocity can be of the form

$$u(x,y)=c_1(x)y^2+c_2(x)y+c_3(x) \tag{8}$$

with boundary conditions

$$u(\cdot,0)=U_0; u(\cdot,h)=0; v(\cdot,0)=0; v(\cdot,h)=0 \tag{9}$$

Applying equations (9) to (8) and (7) gives us the velocity components

$$u = U\left((1-y^2/h^2)+g(x)y(1-y/h)\right) \tag{10}$$

$$v = -Uh'(x)\left(\frac{y^2}{h^2}(2-\frac{c}{h})(1-\frac{y}{h})\right) \tag{11}$$

[1]Closed form solutions of velocity and stresses for some positive integer values of q have also been obtained.

where $g(x)=\frac{c-4h}{h^2}$, h can be freely chosen and c is a constant yet to be determined using stress conditions on the boundary. This value will depend on fluid parameters such as q, k, η_0 and the normal stress coefficients and are recorded in table 1 for different fluid parameters. In figure 3 several different profiles of u and v versus y at different horizontal positions in the channel of varying top plate slopes are provided. As expected, the horizontal component of velocity u has a maximum at the bottom plate which is moving and monotonically decreases with increasing y. The effect of the slope m on u goes into adding some curvature effects; when $m=0$, u varies linearly with y everywhere in the channel. The vertical component of velocity essentially vanishes in the parallel plate case but is non-zero with a flow in the positive y or negative y for diverging or converging channels, respectively. In the former case, there is a net flow downward while in the latter geometry the net flow is pointed upward. Since the exact nature of the slope is hard to estimate in the eye, it is plausible that small vertical flows could come into play as well. Therefore in our study, the cases of $m=0.1, 0$ and -0.1 have also been examined.

Once the equations (10) and (11) are available, we can compute the stresses in the fluid T_{xx}, T_{yy} and T_{xy}. If we additionally assume lubrications conditions, namely

$$v << u; \frac{\partial u}{\partial x} << \frac{\partial u}{\partial y}; \frac{\partial v}{\partial x} << \frac{\partial v}{\partial y} \tag{12}$$

then the linear momentum equations in the x and y directions along with equations (12) can be used to obtained the equations for the stresses on the boundaries (see [13] for a detailed derivation). For the computation of the normal stress, $N=T_{yy}$, we take the boundary conditions $N|_{x=0}=N_{x=1}=0$.

The stresses display variations with respect to several parameters, q, De and m. The essential profile remains the same, however the magnitude can change significantly depending upon the above mentioned parameter values. In the figure 4 we show a sample plot displaying this variation for the shear thinning case $q=-0.001$ and $De=0.25$ for a variety of slopes both positive and negative. In the bottom part of this plot we the effect of De for the parallel channel (i.e. $m=0$) is also shown.

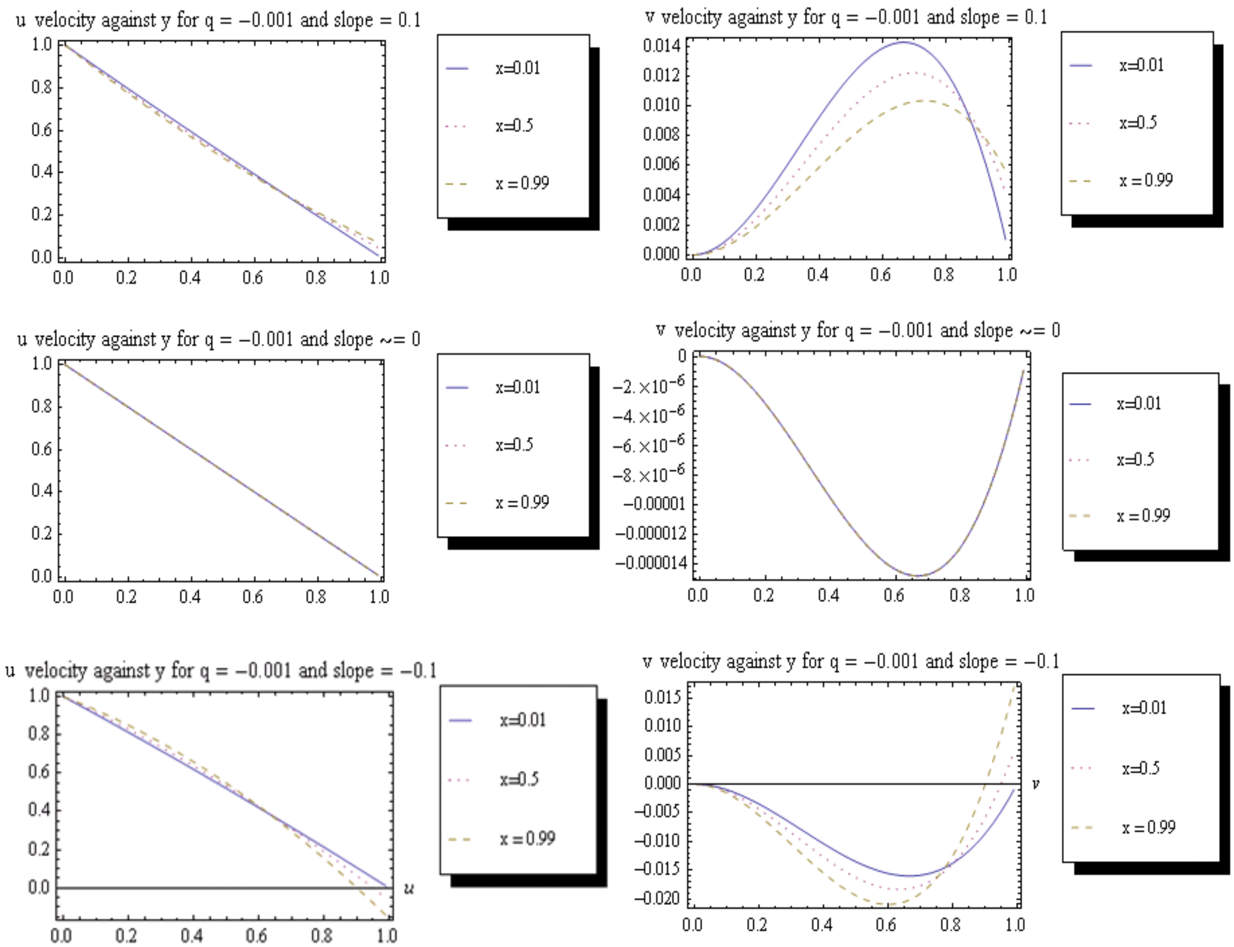

FIGURE 3. PLOTS OF THE HORIZONTAL (LEFT) AND VERTICAL (RIGHT) COMPONENTS OF THE VELOCITY OF A FLUID WITH DEBORAH NUMBER OF 0.25, SLOPE OF -0.2 AND Q=-0.001 AT X=0.01,0.5 AND 0.99 PLOTTED AGAINST Y=0 TO Y=1. IN THE SECOND ROW, THE EXACT VALUE OF THE SLOPE M=-0.0001. AS IS CLEAR FROM THESE PICTURES, THE HORIZONTAL FLOW IS UNIDIRECTIONAL AND DOES NOT HAVE A VERTICAL FLOW COMPONENT. HOWEVER, IN THE CONVERGING AND DIVERGING CASES, THE NET FLOW IS POINTED IN THE DOWNWARD AND UPWARD DIRECTIONS, RESPECTIVELY.

The differences in magnitude of N are not visible in the scale presented. A preliminary test case for $q = 0$ was performed and our results matched perfectly with those of [13] for a second order fluid with constant viscosity. Our overall profile and order of magnitude for N matches with those of other studies in the literature [3, 21, 22] as well. We realize that experimental data on the stresses in the eyelid are hard to come by and have not been able to find any appropriate literature to this end.

We are instead attempting to do our own scaled up experiments to measure stresses imposed on the walls under shearing motions upon a viscoelastic fluid. This part of the project is still in incipient stage and we do not yet have corroborative data to show at this stage. The only relevant experimental paper that we find in the literature is the recent work of Sharma and Pandey ([23]) who, in 2009, experimentally studied the pressure distributions inside a pad thrust bearing for oil (both clean and contaminated) which is a Newtonian liquid. We have been unable to find any such work for viscoelastic, non-Newtonian fluids so far. While the work of Sharma and Pandey cannot be used as evidence for our own work we can certainly examine trends in their study to compare to ours. The primary objective of this experimental study is to compare and contrast shapes of pads in thrust bearings and measurements are made with different pads where the height function $h(x)$ is linear, convex, concave and also a step function. The pressure distribution in the pad is measured with respect to the horizontal distance. The pressure profile measured is parabolic for the linear bearing geometry as is also seen in our case which persists even in the case of vanishing De.

TABLE 1. SOME SAMPLE VALUES OF C FOR DIFFERENT VALUES OF SLOPE M AND DE FOR THE FIXED SHEAR-THINNING CASE OF $Q = -0.001$.

m	De	c
0.1	0,0.25,0.5,0.75	2.84,3.07,3.38,3.73
0.0	0,0.25,0.5,0.75	2.99,3.00,3.0002,3.0004
-0.1	0,0.25,0.5,0.75	3.14,2.96,2.83,2.72

Particulate Matter in the Tear Film

Consider a particle embedded in the eye represented by a sphere of radius 'a'. The motion of such a particle in the sur-

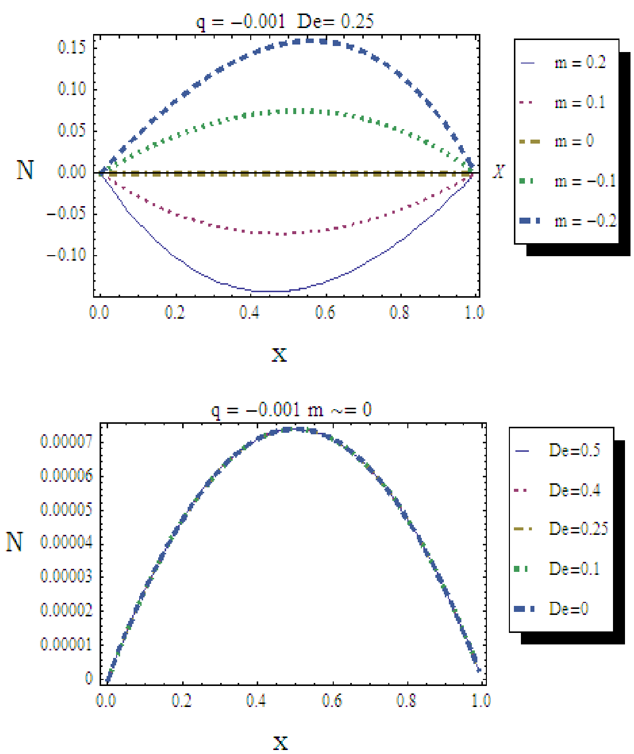

FIGURE 4. THE PLOT SHOWS THE VARIATION IN THE NORMAL STRESS, $N = T_{YY}$, UPON THE BOTTOM PLATE AS A FUNCTION OF X, SLOPE M AND DE. IN THE UPPER FRAME WE FIX $De = 0.25$ AND $Q = -0.001$ AND WE SEE THE STRESS VARIATION IN X WITH THE RESPECT TO THE SLOPE OF THE PLATES. IN THE LOWER FRAME WE SHOW THE VARIATION IN NORMAL STRESS WITH De.

rounding viscoelastic fluid medium can be approximated by the considering the various forces upon it:

$$\rho_p \frac{d\mathrm{v}_p}{dt} = \mathrm{F}_{AM} + \mathrm{F}_B + \mathrm{F}_D + \mathrm{F}_L \tag{13}$$

namely, force due to the added mass effect (F_{AM}), the Bassett force(F_B), drag (F_D) and lift (F_L) forces. The force due to gravity can be ignored in this context. Such an empirical equation has been effectively derived by Wiberg and Smith [27] which is essentially a modified form of the Maxey-Riley equation and is given in non-dimensional form by:

$$S\frac{d\mathrm{v}_p}{dt} = \frac{D\mathrm{v}}{Dt} + \frac{1}{2}\left(\frac{D\mathrm{v}}{Dt} - \frac{d\mathrm{v}_p}{dt}\right) + \frac{3}{4}\frac{C_D}{D}|\mathrm{v} - \mathrm{v}_p|(\mathrm{v} - \mathrm{v}_p)$$

$$+\frac{3}{4}\frac{C_L}{D}(|\mathrm{v}-\mathrm{v}_p|^2_{\text{top}}-|\mathrm{v}-\mathrm{v}_p|^2_{\text{bottom}})\hat{n} \quad (14)$$

where $S=\rho_p/\rho_f$, $\hat{n}$ is the unit vector normal to the flow, v_p represents the particle velocity, D/Dt is the material derivative, and C_D and C_L are the drag and lift coefficients, respectively, v represents the fluid velocity and D is the particle diameter. The drag coefficient is empirically chosen depending on particle Reynolds number, $Re_p=(\rho D|\mathrm{v}-\mathrm{v_p}|)/\mu$ where ρ and μ are the density and viscosity of the fluid, respectively;

$$C_D = \begin{cases} \dfrac{24}{Re_p}, & \text{if } Re_p<1; \quad (15)\\[2ex] \dfrac{24}{Re_p}(1+0.15Re_p^{0.687}) & \text{if } Re_p>1. \quad (16)\end{cases}$$

An accurate determination of C_D requires an evaluation of the above formulae at every time step which is then fed back into the equation (14). For our immediate purposes, we will approximate C_D as a constant. Based on the work of [29], the Re for the tear film is known to be in the range $10^{-7}-10^{-9}$. Assuming a particle moving initially at about half the speed of the flow [2]gives us a particle Reynolds number, Re_p of about the same order of magnitude as the tear film. This gives us $C_D\approx 10^8-10^{10}$. While the different terms on the right hand side of equation (14) are quite clear, the final term regarding the lift needs some explanation. The lift force is given obtained from the following relation,

$$\bar{F}_L=\frac{1}{2}C_L\rho_p|v_p-v|^2A$$

where A is the reference area given by half the surface area of the sphere. While the lift coefficient has a complicated dependence upon the Re, we employ the tabulations of this relationship for a freely rotating sphere due to [18] to obtain an average value of C_L[3]. Based on [18] we take $C_L\approx 1$ based on the fact that the Re_p is sufficiently small. Since the lift on a sphere is caused by the local shear force experienced by the sphere, we further approximate this term by (also see figure 5)

$$\mathrm{F}_L\approx\frac{\bar{F}_L|_{\text{top of sphere}}-\bar{F}_L|_{\text{bottom of sphere}}}{V_{sph}}. \quad (17)$$

[2]It is reasonable to assume that the high viscosity of the mucus layer and its protective nature does not allow for arbitrarily large speeds.

[3]The values of C_D and C_L are still based on their values in a Newtonian fluid. For sake of higher accuracy their dependence upon the De would also be essential.

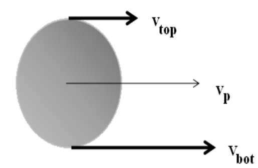

FIGURE 5. A SCHEMATIC OF THE LOCAL SHEAR LAYER IN THE NEIGHBORHOOD OF THE PARTICLE. THE RESULTING SHEARING FORCE INDUCES A COUNTERCLOCKWISE ROTATION THUS GIVING THE PARTICLE A LIFT FORCE.

The equation (14) along with initial conditions

$$x_p(0)=x_0, y_p(0)=y_0; \quad (18)$$
$$x'_p(0)=v_x^{(0)}, y'_p(0)=v_y^{(0)} \quad (19)$$

is solved using a Runge-Kutta method with variable time step. We present results, namely the position of the particle in time $(x_p(t),y_p(t))$ for varying slopes and De. The effect of the slope and De and m are analyzed in figure 6. In the cases when m is zero or negative, the downward vertical flow would induce a downward motion to the particle, towards the moving plate. In the case of the positively sloped plate the magnitude of the vertical velocity is far greater than the downward lift force acting on the particle, therefore the particle undergoes an upward motion away from the moving plate. It is to be noted that even for the parallel plate case, there is variation in the particle trajectory due to De even though it is difficult to tell from this scale used in this figure. Our calculations suggest that there are sufficient reasons for exploring this model as one for the mechanics of the eye even further. At this stage there is no experimental work to compare our work with. We are in the process of conducting our own experimental studies to verify our theoretical results.

CONCLUSIONS

This paper explores the mechanics of a viscoelastic (with shear dependent viscosity) fluid in a slider bearing geometry and the effect of such a flow on an immersed particle. The problem derives inspiration from the physiological nature of the eye with its multiple layering of the tear film. In particular we want to investigate the trajectory of the embedded particle due to the viscoelasticity in the fluid. In the context of the eye, the central question of interest is whether the mucus layer which is very viscous and elastic is strategically placed above the cornea in order to both arrest the penetration of any foreign body and

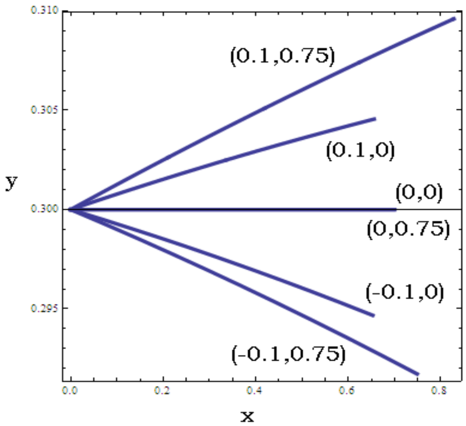

FIGURE 6. THE PLOT SHOWS PARTICLE TRAJECTORIES OVER TIME FOR DIFFERENT M AND De. THE INITIAL VELOCITY OF THE PARTICLE IS TAKEN TO BE HALF THE VELOCITY OF THE FLUID AT TH EINITIAL POSITION, DIAMETER, $D = 0.1$, $S = 1.1$. THE LABELS MENTIONED ALONGSIDE THE TRAJECTORIES INDICATE (M, De).

also to push it away from the cornea. Our preliminary results are obtained under several approximations: a single fluid layer is considered, the walls are taken to be rigid, the normal stress coefficients are taken to be constants, the shear-rate dependent viscosity is taken for sufficiently small values of q of about -0.001 while the appropriate physiological values of $-1 < q < -0.6$ [29] and we consider a simple approximation of the lift force in equation (14). Finally, in addition to the above mentioned considerations a more realistic blinking motion must be replicated i.e. oscillatory motion of the plate.

The problem being addressed is a highly nonlinear system of differential equations, representing the fluid and solid portions under the eyelid. The approach being considered here is definitely non-trivial. The particular novelty of this work is the modeling of the particle by using the Wiberg-Smith equations within a lubrication setting. Our computations shows tendency for an immersed particle to drift to the bottom plate under most circumstances. Since the bottom plate is moving, it represents the eyelid, while the stationary upper plate would represent the eyeball. Thus the motion towards the moving plate would indicate the removal of the particle from the mucus layer of the eye. The only case when the particle gets pushed away from the eyelid is when we take a diverging plate geometry when the vertical flow away from the eyelid dominates. The exact nature of the particle motion is however, dependent on its initial conditions and the material properties of the fluid which need to be examined in greater detail. This is a first study in our eventual goal of obtaining a better understanding the physics of the eye. Our results defintiely show reason to explore the problem in greater detail in the future, accounting for the above mentioned shortcomings.

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