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FLOW OF MOVING WALL JET NEAR CHANNEL EXIT AT MODERATE REYNOLDS NUMBER

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ABSTRACT

The two-dimensional jet flow of a Newtonian fluid at moderate Reynolds Number emerging from a channel where the upper plate is moving is examined theoretically in this study. In this case, the equations of motion are reduced by expanding the flow field about the basic Couette flow. Inertia is assumed to be large enough, allowing asymptotic development in terms of the inverse Reynolds number. A boundary layer forms adjacent to the free surface, and a classical boundary-layer analysis is applied to find the flow in the free surface and the moving wall. The influence of this boundary layer is investigated with the aid of the method of matched asymptotic expansions. The flow and stress fields are obtained as composite expansions by matching the flow in the boundary-layer region near the free surface and the flow both in the inner (boundary-layer) region and in the outer region of the core. The influence of wall velocity on the shape of the free surface, the velocity and stress is emphasized. The formulation allows for the determination of the steady state flow and free surface profiles analytically. The present work provides the conditions near exit, with the help of Higher-order boundary-layer effects (i.e. the cubic term of the inverse Reynolds number), to determine the jet structure further downstream.

INTRODUCTION

The flow configuration corresponds, generically, to a Newtonian jet inside a channel, flowing onto a moving wall as it emerges out of the channel. The flow near the channel exit is closely examined, and the influence of wall velocity is emphasized. Although the moving wall jet problem has its own challenges, it presents common fundamental characteristics with other laminar free surface jet flows which have been studied in the past. The literature abounds on free jets (Tillett [1]; Philippe and Dumargue [2]), impinging jets (Watson [3]; Bowles and Smith [4]; Bush and Aristoff [5]; Phares, Smedley and Flagan [6]), and gravity driven jets (Ruschak and Scriven [7]; Wilson [8]) and to a much lesser extent, wall jets (Elliotis, Georgiou and Xenophontos [9]). See, for instance, the early analyses of Glauert [10] for a jet on a flat wall, Wygnanski and Champagne [11] for a jet on a curved wall, the experimental study of Maki [12] for moving wall jet, and the recent computational analysis and measurements of Levin, Chernoray, Fdahl and Henningson [13] for a jet on a stationary flat wall.

When a free surface jet emerges from a tube or a channel, an abrupt change in stress occurs at the exit. This stress singularity constitutes the major difficulty in any theoretical analysis. In particular, if a computational approach is adopted, the incorporation of the singularity point and its immediate vicinity is unavoidable since the entire flow domain must be considered (discretized). The singularity region, which is crucial to the rest of the flow domain, is difficult to handle numerically if a satisfactory level of accuracy is sought. In contrast, the asymptotic approach lends itself efficiently as a viable alternative. Perhaps more importantly, asymptotics tend to provide deeper insight on the flow structure near the singularity.

Asymptotic analyses tend to circumvent the singularity by identifying two distinct flow regions: a boundary layer region near the free surface, extending but not including the singular point, and a core region where the flow remains close to fully developed. The inclusion of the singularity is not essential in this case given the similarity character of the flow in the boundary layer region. Note again that the boundary layer region extends both upstream and downstream from the singularity. However, although the flow does not remain fully developed as it approaches the exit, the thickness of the boundary layer upstream of the exit is generally small at high Reynolds number, and is often ignored.

Although moving wall jet flow is of fundamental importance, it is also of significant practical relevance. In particular, wall jet flow has been mainly examined in the context of the coating process. Although various coating flow configuration exists in practice, knife coating is the closest relevance to current problem.

Following existing asymptotic analyses in the literature for laminar free surface flows, the method of matched asymptotic expansions is used to examine the moving wall jet at high Reynolds number. Similarly, a classical boundary layer analysis is developed in the present problem near the free surface, and the boundary layer flow is matched to the inviscid flow in the core region. The flow field is thus determined at small distances downstream of the jet. Similarly to all boundary layer analyses, where the solution is not valid within a small distance from inception such as very near a leading edge or a stagnation point (for an impinging jet), the analysis precludes the flow at the channel exit. However, the distance in question is small, on the order of the (local) boundary layer thickness. Consequently, the boundary layer approach turns out to be successful in capturing the flow nature near inception. The solution is developed in powers of ε, where ε^3 is the inverse Reynolds number, both in the "inner" boundary layer region and in the "outer" core region. Special emphasis is placed on the effect of wall velocity on the shape of the free surface and the profile of the velocity close to the exit.

GOVERNING EQUATION

Consider the two-dimensional flow of an incompressible fluid of density ρ and viscosity μ, emerging from a channel of width D. The flow configuration is schematically depicted in figure 1 in the (X, Z) plane.

The X axis is taken along the stationary wall and the Z axis is chosen in the transverse direction across the channel. The channel exit coincides with $X = 0$.The flow is induced by the translation of the upper wall, moving at velocity C.

The stream function of the basic Couette flow is given by

$$\Psi = \frac{C}{D}\frac{z^2}{2} \qquad (2.1)$$

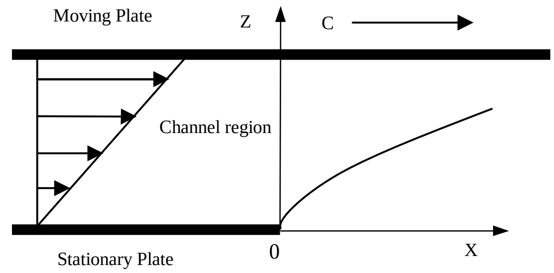

FIGURE 1. SCHEMATIC ILLUSTRATION OF FLOW CONFIGURATION

Non-dimensional variables are introduced by measuring lengths with respect to D and stream function with respect to CD. In this case, the Reynolds number, Re, is given by

$$\mathrm{Re} = \frac{DC}{\nu}$$

Where ν is the kinematic viscosity. Equation (2.1) will turn out to be the leading order solution in the core region, and is conveniently introduced here as

$$\psi_0 = \frac{z_0^2}{2} \qquad (2.2)$$

In this study, Re is assumed to be moderately large. The non-dimensional conservation of momentum equation for the laminar steady flow takes the following form

$$\psi_z\psi_{xz} - \psi_x\psi_{zz} = -p_x + \frac{1}{\mathrm{Re}}\left(\psi_{xxz} + \psi_{zzz}\right) \qquad (2.3a)$$

$$-\psi_z\psi_{xx} + \psi_x\psi_{xz} = -p_z - \frac{1}{\mathrm{Re}}\left(\psi_{xxx} + \psi_{xzz}\right) \qquad (2.3b)$$

For $x > 0$, the kinematic and dynamic boundary conditions at the free surface $z = \zeta(x)$ are

$$\psi(x, z = \zeta) = 0 \qquad (2.4a)$$

$$p + \frac{1}{\mathrm{Re}}\left[2\psi_{xz} + \zeta'\left(\psi_{zz} - \psi_{xx}\right)\right] = 0 \qquad (2.4b)$$

$$p\zeta' - \frac{1}{\mathrm{Re}}\left(2\psi_{xz}\zeta' - \psi_{zz} + \psi_{xx}\right) = 0 \qquad (2.4c)$$

A prime denotes total differentiation. Inside the channel ($x < 0$), the following conditions must be satisfied, namely,

$$\psi_z = 1, \quad \psi_x = 0 \quad at\ z = 1 \qquad (2.5a)$$

$$\psi_z = 0 \quad at\ z = 0 \qquad (2.5b)$$

$$\psi \to \frac{z^2}{2} \; as \;\; x \to -\infty \tag{2.5c}$$

The flow is supposed to have the basic Couette profile Eqn. (2.2) to lowest order and is modified when the fluid leaves the channel in the form of the wall jet. When the fluid detaches itself from the wall of the channel, the removal of the wall stress causes a boundary layer to form in a region near the free surface. In this region, the velocity profile adjusts itself so as to satisfy the condition of zero traction at the free surface. In the inviscid limit, this condition would not be imposed since there is no (viscous) mechanism for the stress singularity to diffuse. However, no uniqueness theorem exists for this inviscid problem, and it is conceivable that other solutions might exist. Nevertheless, it is assumed in this paper that the fully developed Couette flow is everywhere the proper inviscid limit. With this assumption, the flow in the core of the jet is, to lowest order, not affected by the flow in the boundary layer region near the free surface although the boundary layer is expected to induce perturbations to the basic Couette flow, when higher order terms are included, both for the flow upstream and downstream from the channel exit.

In particular, for the flow outside the channel, the region close to the free surface, the inner region, is shear dominated and the flow is of the boundary layer type. In the region between the interface and the moving wall, the outer region, both shear and elongation prevail as a result of the predominance of the Couette character of the flow and the contracting jet. The outer region extends to the channel interior and is part of the core region. At the channel exit, x = 0, the shear stress undergoes a step change from a non-zero value at the lower wall, z = 0, to zero at the free surface, z = ζ(x). The effect of this drop diffuses upstream inside the channel (x < 0) over a distance x_0 where fully developed Couette flow is recovered, and downstream (x > 0) toward the moving wall over a distance x_∞, at which point the flow is entirely of the boundary layer type. The current study focuses on the flow outside the channel where the similarity solution in the inner region is matched onto the outer solution. This latter in turn is matched onto the outer solution in the core region inside the channel at the channel exit. It is important to observe that no matching is required for the similarity solution at x = 0. This constitutes a major advantage of the current formulation compared to alternative solution methods.

The problem is now examined by considering separately the flow near the free surface (inner region), the flow in the core region, and the flow in the vicinity of the moving wall (outer region). The composite flow is obtained upon matching the solutions at the interface between the two regions. Part of the formulation in each layer is similar to the free jet formulation carried out by Tillett [1].

PERTURBATION EQUATIONS

Flow in the Inner Region

The inner expansion for ψ begins with a term in ε^2. Thus, the expansion proceeds in powers of ε so that

$$\psi(\xi,\eta) = \varepsilon^2 \Psi_2(\xi,\eta) + \varepsilon^3 \Psi_3(\xi,\eta) + \cdots \tag{3.1}$$

Similarly, h is expanded as

$$h(\xi) = h_0(\xi) + \varepsilon h_1(\xi) + \varepsilon^2 h_2(\xi) + \cdots \tag{3.2}$$

It is concluded that p is of order ε^4 inside the channel. Thus,

$$p(\xi,\eta) = \varepsilon^4 P_4(\xi,\eta) + \varepsilon^5 P_5(\xi,\eta) + \ldots \tag{3.3}$$

A similarity solution can be carried out for Ψ_2; which is written here as

$$\Psi_2(\xi,\eta) = \xi^{2/3} f_2(\theta) \tag{3.4}$$

Where $\theta = \eta\xi^{-1/3}$ is the similarity variable. The equation for $f_2(\theta)$ is given by

$$3 f_2''' + 2 f_2 f_2'' - f_2'^2 = 0 \tag{3.5}$$

subject to the following boundary conditions:

$$f_2(0) = f_2''(0) = 0 \tag{3.6}$$

$$f_2(\theta) \sim \frac{\theta^2}{2} \;\; \text{as} \quad \theta \to \infty \tag{3.7}$$

For large θ, an asymptotic solution is possible to obtain subject to condition (3.7) following similar arguments, leading to

$$f_2(\theta \to \infty) = \frac{(\theta + c_1)^2}{2} \tag{3.8}$$

Where c_1 is a C dependent constant determined from numerical integration.

To the next order in ε, a linear equation is obtained for Ψ_3, with variable coefficients, admitting a similarity solution of the form Tillett [1]

$$\Psi_3(\xi,\eta) = \xi \, f_3(\theta) \tag{3.9}$$

The equation for $f_3(\theta)$ is given by

$$3f_3''' + 2f_2 f_3'' - 3f_2' f_3' + 3f_2'' f_3 = 0 \qquad (3.10)$$

subject to the following boundary conditions:

$$f_3(0) = f_3''(0) = 0 \qquad (3.11)$$

The third boundary condition is obtained from matching:

$$f_3(\theta) \sim 0 \quad as \quad \theta \to \infty \qquad (3.12)$$

For higher order-boundary layer, proceeding with the latter expansion (ε^6), the equation for $\Psi_4(\xi,\eta)$

$$\Psi_{4\eta\eta\eta} + \Psi_{2\xi}\Psi_{4\eta\eta} + \Psi_{2\eta\eta}\Psi_{4\xi} - \Psi_{2\eta}\Psi_{4\xi\eta} - \Psi_{2\xi\eta}\Psi_{4\eta} = F + G \qquad (3.13)$$

Where

$$F = \Psi_{3\eta}\Psi_{3\xi\eta} - \Psi_{3\xi}\Psi_{3\eta\eta} = \xi^{\frac{1}{3}}\left(\frac{2}{3} f_3'^2 - f_3 f_3''\right) = 0$$

$$G = P_{4\xi} - h_0' P_{4\eta} - \Psi_{2\xi\xi\eta} + h_0''\Psi_{2\eta\eta} + 2h_0'\Psi_{2\xi\eta\eta} - h_0'^2\Psi_{2\eta\eta\eta}$$

The expression obtained here leads us to write $\Psi_4(\xi,\eta)$ in the form

$$\Psi_4(\xi,\eta) = \xi^{\frac{4}{3}} f_{4a}(\theta) + \xi^{-\frac{2}{3}} f_{4b}(\theta) \qquad (3.14)$$

$f_{4a}(\theta)$ and $f_{4b}(\theta)$ then satisfy

$$f_{4a}''' + \frac{2}{3} f_2 f_{4a}'' - \frac{4}{3} f_2' f_{4a}' + \frac{4}{3} f_2'' f_{4a} = \frac{2}{3} f_3'^2 - f_3 f_3'' \qquad (3.15)$$

$$f_{4b}''' + \frac{2}{3} f_2 f_{4b}'' + \frac{2}{3} f_2' f_{4b}' - \frac{2}{3} f_2'' f_{4b} = g \qquad (3.16)$$

Where

$$g = \frac{1}{27} t(4 f_2 f_2' - 2t f_2'^2) + \frac{4}{81}(4 f_2^2 - 2t f_2 f_2' - 3t f_2'' + 3 f_2') - \frac{2}{9}(t f_2'' - f_2' + \frac{1}{2} t^2 f_2''') + \frac{8}{27} f_2'(0)$$

Here $t = \theta + c_1$

As f_3 turns out to be zero, eqn. (3.15) becomes zero. So, eqn. (3.16) subject to the following boundary conditions:

$$f_{4b}(0) = 0,\ f_{4b}''(0) = \frac{4}{9} c f_2'(0) \qquad (3.17)$$

The third boundary condition is obtained from matching:

$$f_{4b} \sim B_{4b} t - \frac{4}{9} f_2'(0) + O(t^{-2}) \qquad (3.18)$$

So, the expression for ψ and h can be obtained inside and outside the channel exit,

$$\psi(\xi,\eta) = \varepsilon^2 \xi^{2/3} f_2(\theta) + \varepsilon^4 \xi^{-2/3} f_{4b}(\theta) + \cdots \qquad (3.19)$$

From matching between inner and core region, it has found that,

$$h_0 = c_1 \xi^{1/3} \text{ and } h_2 = \xi^{-1} B_{4b}$$

So, h can be written as,

$$h(\xi) = c_1 \xi^{1/3} + \varepsilon^2 \xi^{-1} B_{4b} + \cdots \qquad (3.20)$$

Flow in the Core Region

In the core region, which is far from the region near $z = 0$, ψ and p are represented by the following expansions:

$$\psi(x,z) = \psi_0(x,z) + \varepsilon\psi_1(x,z) + \cdots \qquad (3.21)$$

$$p(x,z) = p_0(x,z) + \varepsilon p_1(x,z) + \cdots \qquad (3.22)$$

ψ_m ($m > 0$) are higher order terms that denote the deviation from the basic flow due to its interaction with the boundary layer.

To leading order $p_0(x, z) = 0$. For $m = 1$ and 2, the matching conditions and the condition $\psi_{m>0}(x \to -\infty, z) = 0$, lead to the vanishing of the stream function and pressure everywhere. More explicitly,

$$\psi_1(x, z) = \psi_2(x, z) = p_1(x, z) = p_2(x, z) = 0 \qquad (3.23)$$

To next order, $m = 3$, from matching it is obtained that $\psi_3(x, z) = 0$

Noting that $w_3 = -\psi_{3x}$, and using boundary value problem in the ranges $-\infty \le x \le \infty$ and $0 \le z \le 1$ is concluded:

$$\nabla^2 w_3 = 0 \qquad (3.24)$$

$$w_3(x,1) = 0$$

$$w_3(x,0) = 0 \qquad for \quad x < 0$$

$$w_3(x, z \to 0) = 0 \qquad for \quad x > 0$$

$$w_3 \quad \text{bounded as } |x| \to \infty$$

The condition $w_3(x>0, z\to 0)=0$ is obtained from matching. So far, the formulation in this section has been common to both the regions inside and outside the channel. Although the flow fields in these two regions will have to match at the channel exit (x = 0), they can be conveniently examined separately.

Flow in the Outer Region

The outer expansion for ψ and p are represented by the following perturbation expansions

$$\psi(\xi,\eta)=\Psi_0(\xi,\eta)+\varepsilon\Psi_1(\xi,\eta)+\cdots \qquad (3.25)$$

$$p(\xi,\eta)=P_0(\xi,\eta)+\varepsilon P_1(\xi,\eta)+\cdots \qquad (3.26)$$

To leading order, the solution for Ψ_0 is,

$$\Psi_0(\xi,\eta)=\frac{1}{2} \quad P_{0\eta}(\xi,\eta)=0 \qquad (3.27)$$

To the next order, from matching the solution for Ψ_1 is,

$$\Psi_1(\xi,\eta)=-\eta \quad P_{1\eta}(\xi,\eta)=0 \qquad (3.28)$$

RESULTS AND DISCUSSIONS

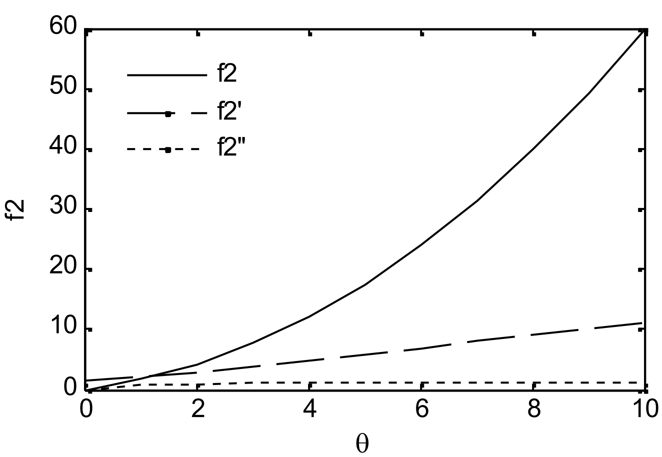

FIGURE 2: VARIATION OF SIMILARITY FUNCTION f_2 WITH SIMILARITY VARIABLE θ

Figure 2 explains that as the problem (3.5) is solved as an initial-value problem, where the equation is integrated subject to the boundary conditions in eqn. (3.6) and (3.7) guessed value of the slope at the origin. The slope is adjusted until reasonable matching is achieved between the solution and the asymptotic form in eqn. (3.8) at large θ. The integration is carried out over the domain $(0,\theta)$.

Similarly figure 3 shows the f_{4b} profiles with similarity variable, θ. Equation (3.16) is integrated subject to the boundary conditions in eqn. (3.17) guessed value of the slope at the origin and the value of B_{4b} is approximately equal to 1.0496.

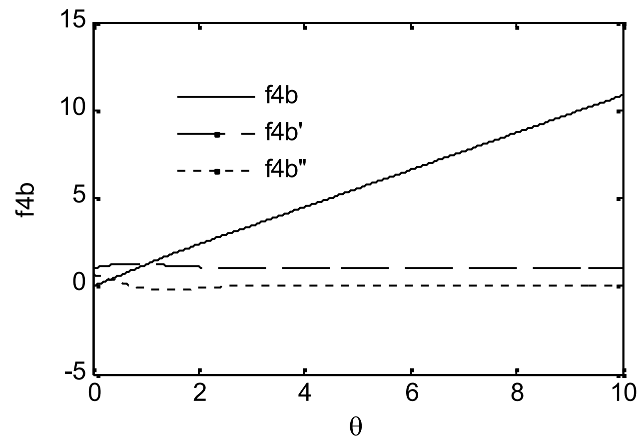

FIGURE 3: VARIATION OF SIMILARITY FUNCTION f_{4b} WITH SIMILARITY VARIABLE θ

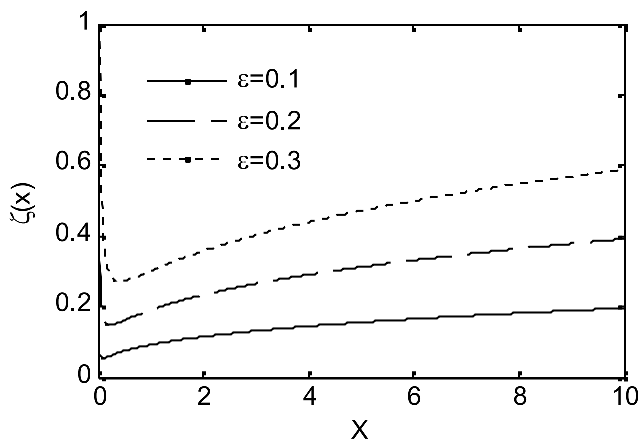

FIGURE 4: FREE SURFACE HEIGHT ζ(x) VERSUS POSITION X AT DIFFERENT ε

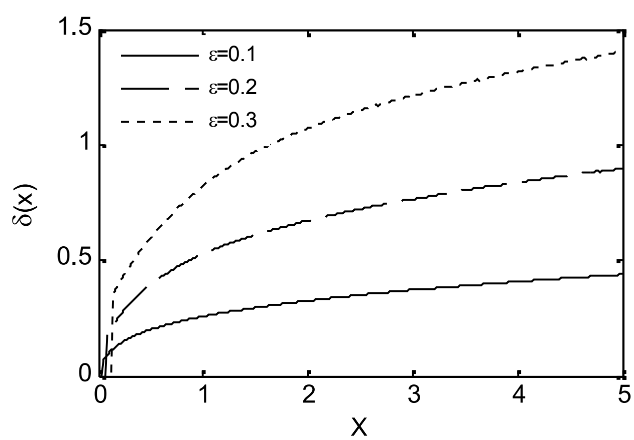

FIGURE 5: DEPENDANCE OF BOUNDARY LAYER δ(x) ON DIFFERENT ε

In figure 4, free surface height is plotted at different x position for different ε. As $\varepsilon = Re^{-\alpha}$, where α= 1/3; therefore Reynolds number decreases with the increases of ε. From the figure, it is seen that outside the channel the free surface contracts as Reynolds number decreases.

Figure 5 shows the change in boundary layer thickness with different ε. As ε increases, the thickness of boundary layer increases.

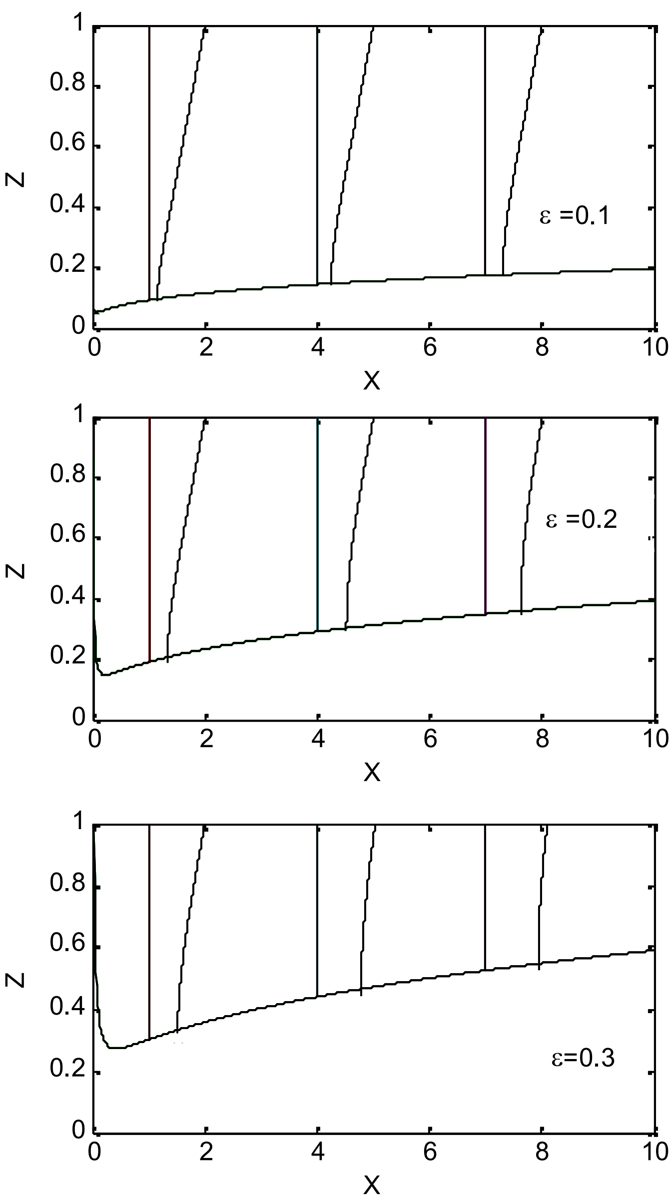

FIGURE 6: STREAM WISE VELOCITY VERSUS POSITION X FOR ε = 0.1, 0.2, and 0.3

Figure 6 exhibits the variation of stream wise velocity at ε = 0.1, 0.2, 0.3 as x increases. The figure is drawn by composite matching between inner-core and outer-core region. The outer-core region gives Couette profile while the matching between inner-core deviates due to the free surface as there is zero traction force available. As Reynolds number decreases, the inertia force also decreases. So, the flow outside the channel becomes more dependent on the velocity of the moving wall while ε increases.

CONCLUSION

The two-dimensional wall jet flow of a Newtonian fluid emerging from a channel and adhering to a moving wall is examined in this study. In this study a special care is given to the stress singularity in the jet flow. Inertia is assumed to be large enough, allowing asymptotic development in terms of the inverse Reynolds number. In this case, the equations of motion are reduced by expanding the flow field about the basic Couette flow. In fact, the very reason for the current successful analysis is the fact that the fully developed flow is a solution to the governing equations at infinite Reynolds number since the solution is not unique in this limit. A classical boundary layer analysis is applied to find the flow adjacent to the free surface where a boundary layer forms for moderate distances downstream from the channel exit. The influence of this boundary layer is investigated by the aid of the method of matched asymptotic expansions. The boundary layer structure near the free surface was examined.

Finally, the significance of the current study and the advantages of the proposed formulation cannot be overstated. In typical jet flow calculations in the literature, fully developed conditions are assumed at inception. The present work provides the correct conditions near exit, which are required to determine the jet structure further downstream. If the jet becomes thin far downstream, a boundary layer formulation can be used with the presently predicted boundary conditions for steady and possibly transient flows (Khayat and Welke [14] , Muhammad and Khayat [15]).

Typically, fully developed (uniform flow) conditions are assumed (Phan Thien, Jin and Tanner [16]). The current formulation also allows for the determination of the steady state flow and free surface profiles analytically. The availability of the steady state in analytical form constitutes a significant advantage for a linear stability analysis on the jet Soederberg [17], and, as often is the case, when the steady state is taken as the initial condition for a transient analysis. The accuracy of initial conditions is crucial, for instance, for a thin jet given the hyperbolicity of the problem (Khayat and Kim [18]).

REFERENCES

[1] Tillett, J. P. K., 1968, "On the Laminar Flow in a Free Jet of Liquid at High Reynolds Numbers," J. Fluid Mech., 32, pp. 273-292.

[2] Philippe, C. And Dumargue, P., 1991, "Étude de l'établissement d'un Jet Liquide Laminaire émergeant d'une Conduite Cylindrique Verticale Semi-infinie et Soumis à l'influence de La Gravité, " J. Appl. Math. Phys. (ZAMP), 42, pp. 227.
[3] Watson, E., 1964, "The Spread of a Liquid Jet Over a Horizontal Plane," J. Fluid Mech., 20, pp. 481.
[4] Bowles, R. I. And Smith, F. T., 1992, "The Standing Hydraulic Jump: Theory, Computations and Comparisons with Experiments," J. Fluid Mech., 242, pp. 147.
[5] Bush, J. W. M. And Aristoff, J. M., 2003, "The Influence of Surface Tension on the Circular Hydraulic Jump," J. Fluid Mech., 489, pp. 229.
[6] Phares, D. J., Smedley, G. T. and Flagan, R. C., 2000, "The Wall Shear Stress Produced by the Normal Impingement of a Jet on a Flat Surface," J. Fluid Mech., 418, pp. 351
[7] Ruschak, K. J. and Scriven, L. E., 1977, "Developing Flow on a Vertical Wall," J. Fluid Mech., 81, pp. 305.
[8] Wilson, D. E., 1986, "A Similarity Solution for Axis-symmetric Viscous-Gravity Jet," Phys. Fluids, 29(3), pp. 632.
[9] Elliotis, M., Georgiou, G. and Xenophontos, C., 2005, "Solution of the Planar Newtonian Stick–slip Problem with the Singular Function Boundary Integral Method," Int. J. Num. Meth. Fluids, 48, pp. 1001.
[10] Glauert, M. B., 1956, "The Wall Jet," J. Fluid Mech., 1, pp. 625
[11] Wygnanski, I. J. and Champagne, F. H., 1964. "The Laminar Wall-Jet Over a Curved Surface," J. Fluid Mech., 31, pp. 459.
[12] Maki, H., 1983, "Experimental Studies on the Combined Flow Field Formed by a Moving Wall and a Wall Jet Running Parallel to it," Bullt. Japan Soc. Mech Eng., 26, pp. 2100.
[13] Levin, O., Chernoray, V. G., Fdahl, L. L. and Henningson, D. S., 2005, "A Study of the Blasius Jet," J Fluid Mech., 539, pp. 313.
[14] Khayat, R. E. and Welke, S., 2001, "Influence of Inertia, Gravity, and Substrate Topography on the Two-Dimensional Transient Coating Flow of a Thin Newtonian Fluid Film," Phys. Fluids, 13, pp. 355.
[15] Muhammad, T. and Khayat, R. E., 2004, "Effect of Substrate Movement on Shock Formation in Pressure-Driven Coating Flow," Phys. Fluids, 13, pp. 355.
[16] Phan-Thien, N., Jin, H. and Tanner, R. I., 1989, "On the Hydrodynamic Braking Flow of a Viscoelastic Fluid," Wear, 133, pp. 323.
[17] Soederberg, L. D., 2003, "Absolute and Convective Instability of a Relaxational Plane Liquid Jet," J. Fluid Mech., 493, pp. 89.
[18] Khayat, R. E. and Kim, K., 2002, "Influence of Initial Conditions on Transient Two-Dimensional Thin-Film Flow," Phys. Fluids, 14, pp. 4448.

