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FLOW OF WALL JET NEAR CHANNEL EXIT AT MODERATE REYNOLDS NUMBER

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ABSTRACT

The wall jet flow near channel exit at moderate Reynolds Number, emerging from a two-dimensional channel, is examined theoretically in this study. Poiseuille flow conditions are assumed to prevail far upstream from the exit. The problem is solved using the method of matched asymptotic expansions. The small parameter involved in the expansions is the inverse Reynolds number. The flow and stress fields are obtained as composite expansions by matching the flow in the boundary-layer region near the free surface, flow in the outer layer region and the flow in the core region. The fluid is assumed to be Newtonian and it is found that the jet contracts downstream from the channel exit. The influence of inertia on the shape of free surface, the velocity and stress is emphasized and the higher order boundary layer is explored. To leading order, the problem is similar to the case of the free jet (Tillett) [1] with different boundary conditions. A similarity solution can be carried out using a similarity variable problem which is then solved as an initial-value problem, where the equation is integrated subject to the boundary conditions and a guessed value of the slope at the origin. The slope is adjusted until reasonable matching is achieved between the solution and the asymptotic form at large θ. The level of contraction is essentially independent of inertia, but the contraction moves further downstream with increasing Reynolds number. The present work provides the correct conditions near exit, which are required to determine the jet structure further downstream. If the jet becomes thin far downstream, a boundary layer formulation can be used with the presently predicted boundary conditions for steady and possibly transient flows.

INTRODUCTION

The stress singularity constitutes the major difficulty in any theoretical analysis. In particular, if a computational approach is adopted, the incorporation of the singularity point and its immediate vicinity is unavoidable. The singularity region, which is crucial to the rest of the flow domain, is difficult to handle numerically if a satisfactory level of accuracy is sought. In contrast, the asymptotic approach lends itself efficiently as a viable alternative. Perhaps more importantly, asymptotic tend to provide deeper insight on the flow structure near the singularity. In the current study, the interplay between driving pressure and the stationary wall is examined for the two-dimensional steady jet of an incompressible fluid near channel exit. The flow configuration corresponds, generically, to a pressure driven wall jet inside a channel. The flow near the channel exit is closely examined, and the influence of inertia is emphasized. Inertia is assumed to remain relatively important, allowing the asymptotic development of the flow field in terms of the inverse Reynolds number. The driving pressure is assumed to be dominant. In the current work, Tillett's work is extended to cover more comprehensively the flow close to and upstream from the exit. It is important to observe that, in typical jet flow calculations in the literature, Poiseuille conditions are assumed at inception. The literature abounds on free jets (Tillett[1], Philippe & Dumargue [2]), impinging jets (Watson [3], Bowles & Smith [4]; Bush & Aristoff [5]; Phares, Smedley & Flagan [6]), gravity driven jets (Ruschak & Scriven [7]; Wilson [8]) and, to a much lesser extent, wall jets (Elliotis, Georgiou & Xenophontos [9]), the experimental study of Maki [10] for moving wall jet, and the recent computational analysis and measurements of Levin, Chernoray, Fdahl & Henningson [11] for a jet on a stationary flat wall.

Free surface and interfacial flows are inherently complicated because of the unknown position of the surface or interface. The presence of the stress singularity adds considerably to the complexity of the problem and solution. Both analytical and computational solution methodologies have been pursued in the literature. Although numerical methods seem to have prevailed over analytical approaches for most flow problems, this is not the case for flows with singularity. A combination of analytical and numerical treatments has also been proposed (Shi, Breuer & Durst [12]). In a computational approach, the entire flow domain must be discretized, including the singularity and its surrounding region, both upstream and downstream from the exit. Higher accuracy is achieved through mesh refinement, which captures more effectively the singularity but leads simultaneously to the presence of stronger flow gradients that are difficult to handle numerically (Pasquali & Scriven [13]). In order to circumvent the difficulty with the unknown free surface, Tsukiji & Takahashi [14] wrote the flow equations in a curvilinear coordinate system related to the network comprising the streamlines and their orthogonal trajectories. Although this approach simplifies the implementation of the boundary conditions, it complicates the flow equations.

Asymptotic analyses tend to circumvent the singularity by identifying three distinct flow regions: a boundary layer region near the free surface, extending but not including the singular point, the core region where the flow remains close to fully developed and in the upper wall i.e. the outer region. For a recent perspective on asymptotic analyses, their applications and historic development, the reader is referred to the book by Sobey [15] on interactive boundary layer. A classical boundary layer analysis is developed in the present problem near the free surface, and the boundary layer flow is matched to the inviscid flow in the core region. Similarly the matching between the outer and core region is done. The flow field is thus determined at small distances downstream of the jet. The solution is developed in powers of ε, where ε^3 is the inverse Reynolds number, both in the boundary layer region and in the core region. Special emphasis is placed on the effect of inertia on the shape of the free surface and the profiles of the velocity and stress components close to the exit. Good qualitative agreement is found with measurements, observations and numerical predictions in the literature whenever available. Asymptotic analyses have also been successfully implemented for non-Newtonian flows. See, for instance, the work of Denier & Dabrowski [16] on boundary layer flow, and the work of Zhao & Khayat [17] for the spreading of a liquid jet.

The present work provides the correct conditions near exit, which are required to determine the jet structure further downstream. Finally, the approach can be generalized to tackle other high-Reynolds number laminar flow processes of important practical interest near channel or pipe exits.

GOVERNING EQUATIONS AND BOUNDARY CONDITIONS

Consider the two-dimensional flow of an incompressible fluid of density ρ and viscosity μ, emerging from a channel of width D. The flow configuration is schematically depicted in figure 1 in the (X, Z) plane. The X axis is taken along the stationary wall and the Z axis is chosen in the transverse direction across the channel. The channel exit coincides with $X = 0$.

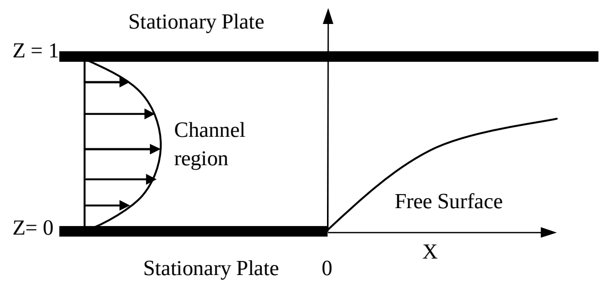

FIGURE 1: SCHEMATIC ILLUSTRATION OF TWO DIMENSIONAL WALL JET

The flow is induced by a pressure gradient, dP/dX, inside the channel. The stream function of the basic Poiseuille flow is obtained from

$$\Psi = \frac{1}{2\mu}\frac{dP}{dX}\left(\frac{Z^3}{3} - D\,\frac{Z^2}{2}\right) = -\frac{6V}{D^2}\left(\frac{Z^3}{3} - D\,\frac{Z^2}{2}\right) \qquad (2.1)$$

where $V = -\dfrac{1}{12\mu}\dfrac{dP}{dX}D^2$ is the mean velocity due to the pressure gradient inside the channel. In this case, V is assumed to be always positive and will be used as the velocity scale. In other words, the pressure gradient is assumed to be always present and negative. Non-dimensional variables are introduced by measuring lengths with respect to D, stream function with respect to VD, and pressure with respect to ρV^2. In this case, dimensionless groups emerge in the problem, namely, the Reynolds number. Now, Eqn. (2.1) will turn out to be the leading order solution in the outer region, and is conveniently introduced here as

$$\psi_0 = 3z^2 - 2z^3 \qquad (2.2)$$

In this study, Re is assumed to be moderately large. The non-dimensional conservation of momentum equation for the laminar steady flow takes the following form

$$\psi_z \psi_{xz} - \psi_x \psi_{zz} = -p_x - \frac{1}{\mathrm{Re}}\left(\psi_{xxz} + \psi_{zzz}\right) \quad (2.3a)$$

$$-\psi_z \psi_{xx} + \psi_x \psi_{xz} = -p_z - \frac{1}{\mathrm{Re}}\left(\psi_{xxx} + \psi_{xzz}\right) \quad (2.3b)$$

For $x > 0$, the kinematic and dynamic boundary conditions at the free surface, $z = \zeta(x)$, are

$$\psi = 0 \quad (2.4a)$$

$$p + \frac{1}{\mathrm{Re}}\left[2\psi_{xz} + \zeta'\left(\psi_{zz} - \psi_{xx}\right)\right] = 0 \quad (2.4b)$$

$$p\zeta' - \frac{1}{\mathrm{Re}}\left(2\psi_{xz}\zeta' - \psi_{zz} + \psi_{xx}\right) = 0 \quad (2.4c)$$

A prime denotes total differentiation. Inside the channel (x < 0), the following conditions must be satisfied, namely,

$$\psi_z = 0 \text{ and } \psi_x = 0 \quad \text{at } z = 1, \quad (2.5a)$$

$$\psi_z = 0 \quad \text{at } z = 0 \quad (2.5b)$$

$$\psi \to 3z^2 - 2z^3 \quad \text{as } x \to -\infty \quad (2.5c)$$

The flow is supposed to have the basic Poiseuille profile (2.2) to lowest order and is modified when the fluid leaves the channel in the form of the wall jet. When the fluid detaches itself from the wall of the channel, the removal of the wall stress causes a boundary layer to form in a region near the free surface. In this region, the parabolic velocity profile adjusts itself so as to satisfy the condition of zero traction at the free surface. In the inviscid limit, this condition would not be imposed since there is no (viscous) mechanism for the stress singularity to diffuse, and all the conditions of the problem would be satisfied by postulating that the parabolic profile continues unchanged in the jet region. However, no uniqueness theorem exists for this inviscid problem, and it is conceivable that other solutions might exist. Nevertheless, it is assumed in this paper that the fully developed Poiseuille flow is everywhere the proper inviscid limit. With this assumption, the flow in the core of the jet is, to lowest order, not affected by the flow in the boundary layer region near the free surface although the boundary layer is expected to induce perturbations to the basic Poiseuille flow, when higher order terms are included, both for the flow upstream and downstream from the channel exit. This assumption is similar to the one made by Smith [18] for the tube flow with severe constriction, where the flow field in the core region, to leading order, satisfy the inviscid equations of motion.

The outer region extends to the channel interior and is different of the core region. At the channel exit, $x = 0$, the shear stress undergoes a step change from a non-zero value at the lower wall, $z = 0$, to zero at the free surface, $z = \zeta(x)$. The effect of this drop diffuses upstream inside the channel ($x < 0$) over a distance x_0 where fully developed Poiseuille flow is recovered, and downstream ($x > 0$) toward the stationary wall over a distance x_∞, at which point the flow is entirely of the boundary layer type. The current study focuses on the flow outside the channel where the similarity solutions in the inner and outer region are separately matched with the core solution. It is important to observe that no matching is required for the similarity solution at $x = 0$, and the flow singularity at the origin is entirely avoided in the solution process. This constitutes a major advantage of the current formulation compared to alternative solution methods.

The problem is now examined by considering separately the flow near the free surface (inner region) and the flow in the core (outer) region and flow in the outer region. The composite flow is obtained upon matching the solutions at the interface between the three regions. Part of the formulation in each layer is similar to the free jet formulation carried out by Tillett [1].

PERTURBATION EQUATIONS

The inner expansion for ψ begins with a term in ε^2 and thus, represented by the following perturbation expansions

$$\psi(\xi,\eta) = \varepsilon^2 \Psi_2(\xi,\eta) + \varepsilon^3 \Psi_3(\xi,\eta) + \cdots \quad (3.1)$$

$$p(\xi,\eta) = \varepsilon^4 P_4(\xi,\eta) + \varepsilon^5 P_5(\xi,\eta) + \cdots \quad (3.2)$$

To leading order, the momentum equation, Eqn. (2.3a) reads

$$\Psi_{2\eta}\Psi_{2\xi\eta} - \Psi_{2\xi}\Psi_{2\eta\eta} = \Psi_{2\eta\eta\eta} \quad (3.3)$$

A similarity solution can be carried out for Ψ_2, which is written here as

$$\Psi_2(\xi,\eta) = \xi^{2/3} f_2(\theta) \quad (3.4)$$

where $\theta = \eta\xi^{-1/3}$ is the similarity variable. The equation for $f_2(\theta)$ is given by

$$3f_2''' + 2f_2 f_2'' - f_2'^2 = 0 \quad (3.5)$$

subject to the following boundary conditions

$$f_2(0) = f_2''(0) = 0 \quad (3.5a)$$

$$f_2(\theta) \sim 3\theta^2 \text{ as } \theta \to \infty \tag{3.5b}$$

An equation similar to (3.3) was investigated by Goldstein [19] and revisited by Tillett [1]. For large θ, an asymptotic solution is possible to obtain, subject to condition (3.5b), namely

$$f_2(\theta \to \infty) = 3(\theta + c_1)^2 + o\left[\exp\left(-\frac{2}{3}\theta^3\right)\right] \tag{3.6}$$

where c_1 is a constant determined from the numerical integration. To the next order from the momentum equation the following equation is found

$$3 f_3''' + 2 f_2 f_3'' - 3 f_2' f_3' + 3 f_2'' f_3 = 0 \tag{3.7}$$

subject to the following boundary conditions:

$$f_3(0) = f_3''(0) = 0 \tag{3.7a}$$

$$f_3(\theta) \sim -2\theta^3 \text{ as } \theta \to \infty \tag{3.7b}$$

An asymptotic solution similar to but more complicated than the case of free jet flow (Tillett [1]) is possible. Thus,

$$\begin{aligned} f_3(\theta \to \infty) &= -2\left[(\theta + c_1)^3 - 1\right] + c_2(\theta + c_1) \\ &+ O\left[\exp\left(-2\theta^3\right)\right] \end{aligned} \tag{3.8}$$

The numerical integration of equation (3.7) gives the value c_2.

In the core region, which is far from the region near z = 0, ψ and p are represented by the following expansions:

$$\psi(x,z) = \psi_0(x,z) + \varepsilon\psi_1(x,z) + \cdots \tag{3.9a}$$

$$p(x,z) = p_0(x,z) + \varepsilon p_1(x,z) + \cdots \tag{3.9b}$$

The solution obtained from the core region to order ε^2 and ε^3 are explicitly,

$$\psi_1(x, z) = \psi_2(x, z) = p_1(x, z) = p_2(x, z) = 0 \tag{3.10}$$

In the core region inside the channel the shape functions V_n are governed by the following Eigenvalue problem:

$$V_n'' + \left(\beta_n^2 + \frac{2}{z - z^2}\right)V_n = 0 \tag{3.11}$$

Downstream from the channel exit, the solution is given as

$$\begin{aligned} w_3(x > 0, z) &= -\psi_{3x}(x > 0, z) \\ &= -2V_0(z) + \sum_{n=1}^{\infty} A_n e^{-\beta_n x} V_n(z) \end{aligned} \tag{3.12}$$

where $V_0(z)$ satisfies the following equation and boundary conditions

$$V_0'' + \frac{2}{z - z^2}V_0 = 0 \tag{3.13}$$

$$V_0(0) = 1 \text{ and } V_0(1) = 0 \tag{3.13a}$$

The outer expansion for ψ and p are represented by the following perturbation expansions

$$\psi(\xi,\eta) = \Psi_0(\xi,\eta) + \varepsilon\Psi_1(\xi,\eta) + \cdots \tag{3.14a}$$

$$p(\xi,\eta) = P_0(\xi,\eta) + \varepsilon P_1(\xi,\eta) + \cdots \tag{3.14b}$$

To leading order, the solution for Ψ_0 is,

$$\Psi_0(\xi,\eta) = 1 \text{ and } P_{0\eta}(\xi,\eta) = 0 \tag{3.15}$$

To order ε^2 from the matching between outer and core it is found that the solution for Ψ_2 is,

$$\Psi_2(\xi,\eta) = 0 \text{ and } P_{2\eta}(\xi,\eta) = 0 \tag{3.16}$$

Similarly for order ε^3 the solution for Ψ_3 is,

$$\Psi_3 = 2x \tag{3.17}$$

THE COMPOSITE FLOW

We obtain the composite velocity profile by matching between the inner and core and between outer and core. Following Van Dyke (1964)[20], the composite expansion operator is defined by

$$C_n \equiv \left(E_n + H_n - E_n H_n\right) \tag{3.18}$$

Here, E_n is the outer-expansion operator, which truncates immediately after the term of order ε^n where the expansion is expressed in terms of outer variables. H_m is the corresponding inner-expansion operator. After successful matching the composite velocity that we obtain is ,

$$C_2 u(x,z) = \varepsilon x^{1/3} f_2'(\theta) + \varepsilon^2 x^{2/3}\left[f_3'(\theta) - 2zc_2 \right] + O\left(\varepsilon^3\right) \quad (3.19)$$

RESULTS AND DISCUSSION

Problem (3.5) is solved as an initial-value problem, where the equation is integrated subject to conditions (3.5a) and a guessed value of the slope at the origin. The slope is adjusted until reasonable matching is achieved between the solution and the asymptotic form (3.6) at large θ. The integration is carried out over the domain $[0,\theta_\infty]$ where θ_∞ is a relatively large value of θ where matching is secured to within an imposed tolerance. Of particular interest here is the value of the slope at the origin, $f_2'(0)$which is directly related to the velocity at the free surface.

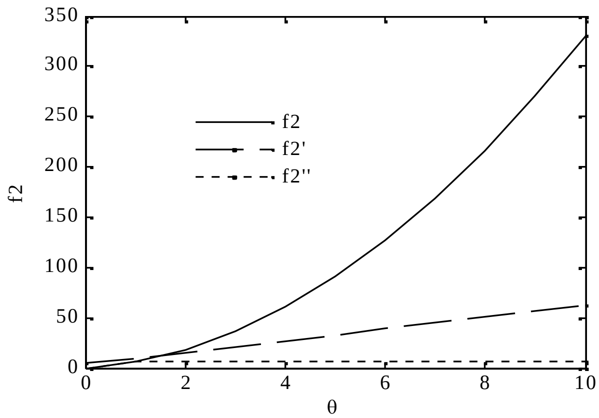

FIGURE 2: VARIATION OF SIMILARITY FUNCTION f_2 VERSUS SIMILARITY VARIABLE Θ

The numerical integration of equation (3.7) gives the value c_1. The solution procedure is similar to before, except that both problems (3.5) and (3.7) are solved as a coupled system.

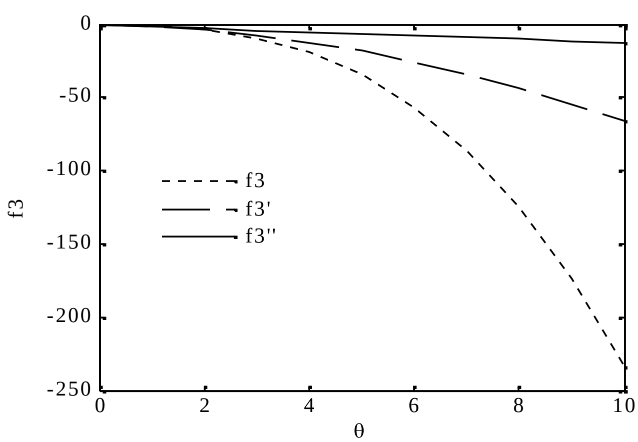

FIGURE 3: VARIATION OF SIMILARITY FUNCTION f_3 VERSUS SIMILARITY VARIABLE Θ

Figure 4 typically illustrates the inner V_n dependence of the Eigen values. Interestingly, V_n decreases and β_n increases with n, making convergence reasonably achievable after only a few modes.

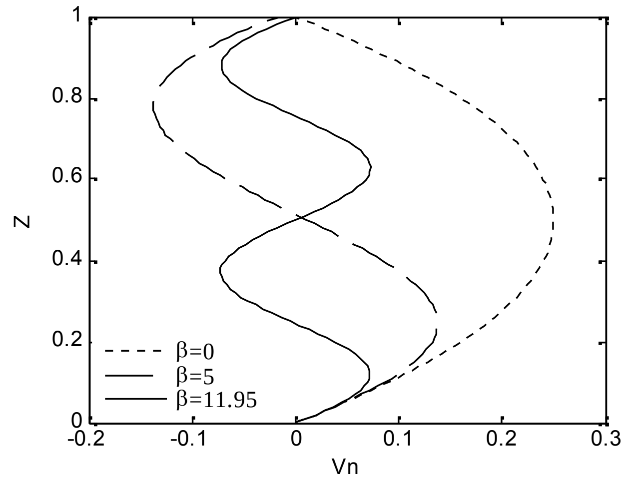

FIGURE 4: SHAPE FUNCTION V_n VERSUS Z

Similarly, solving equation (3.11), figure 5 illustrates the flow in the core (outer) region outside the channel.

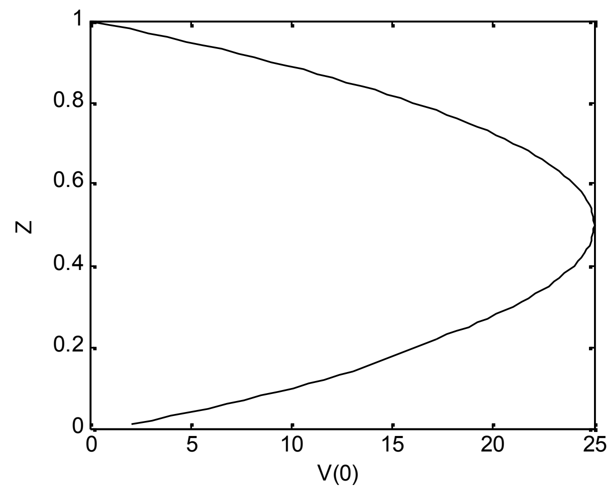

FIGURE 5: SHAPE FUNCTION V_0 VERSUS Z

By using the composite matching between the inner and core and outer and core, the composite velocity is obtained. Figure 6 and 7 represents the outer-core and inner-core composite velocity profile along with the free surface at different position of x for Reynolds No. 1000 and 125 respectively.

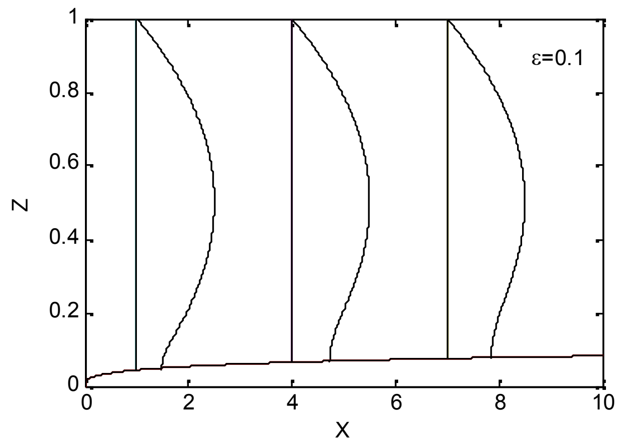

FIGURE 6: COMPOSITE VELOCITY VERSUS Z AT DIFFERENT POSITION OF X AT ϵ=0.1

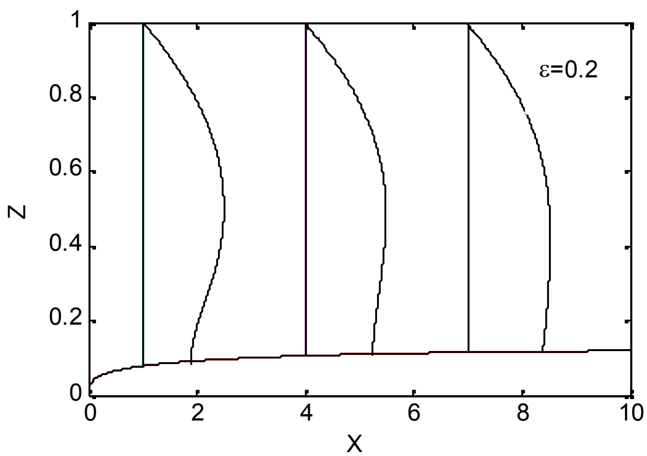

FIGURE 7: COMPOSITE VELOCITY VERSUS Z AT DIFFERENT POSITION OF X AT ϵ=0.2

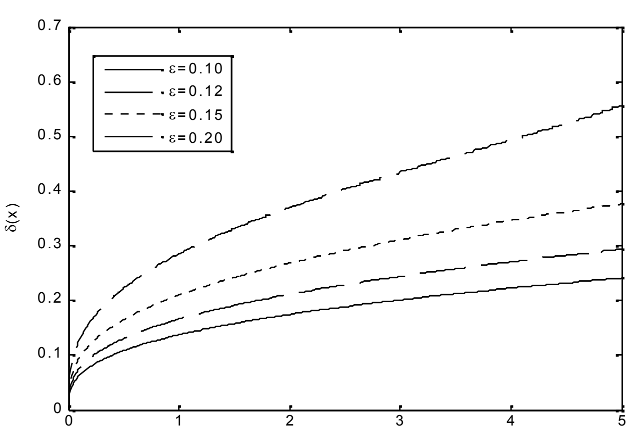

FIGURE 8: BOUNDARY LAYER THICKNESS VERSUS X

The free surface velocity increases with the decrease of Reynolds Number. Figure 8 depicts the thickness of the boundary layer at different ϵ. As Reynolds no. is inversely related to ϵ, it is seen from the figure that with the decrease of Reynolds Number, the boundary layer thickness increases.

CONCLUSION

The two-dimensional wall jet flow of a Newtonian fluid emerging from a channel and adhering to a moving wall is examined in this study. Inertia is assumed to be large enough, allowing asymptotic development in terms of the inverse Reynolds number. In this case, the equations of motion are reduced by expanding the flow field about the basic Poiseuille flow. In fact, the very reason for the current successful analysis is the fact that the fully developed flow is a solution to the governing equations at infinite Reynolds number since the solution is not unique in this limit. A classical boundary layer analysis is applied to find the flow adjacent to the free surface where a boundary layer forms for moderate distances downstream from the channel exit. The influence of this boundary layer is investigated by the aid of the method of matched asymptotic expansions. The roles of inertia of the incoming jet with stationary wall are emphasized.

Finally, the significance of the current study and the advantages of the proposed formulation cannot be overstated. In typical jet flow calculations in the literature, fully developed conditions are assumed at inception. The present work provides the correct conditions near exit, which are required to determine the wall jet structure further downstream.

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