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CAPILLARY INSTABILITY OF LIQUID JETS ISSUING FROM ELLIPTIC NOZZLES

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ABSTRACT

Breakup of a liquid jet issuing from an orifice is one of the classical problems in fluid dynamics due to its theoretical and practical importance. The main application of the process is in spray and droplet formation, which is of interest in the combustion in liquid-fuelled engines, ink-jet printers, coating systems, medical equipment, and irrigation device. The complexity of the breakup mechanism is due to the large number of parameters involved such as the design of injection nozzle, and thermodynamic states of both liquid and gas. In addition, different combinations of surface tension, inertia, and aerodynamic forces acting on the jet, define main breakup regimes. Effects of nozzle geometry on the behavior of liquid jets have been overlooked in the literature. Elliptic jets have never been investigated theoretically since mostly circular jets or liquid sheets have been analyzed; while experiments have shown that by using elliptical nozzles, entrainment and air mixing of fuel in combustion will be increased.

In this article, instability of an elliptic liquid jet under the effect of inertia, viscous, and surface tension forces has been studied using temporal linear analyses. The effects of the gravity and the surrounding gas have been neglected. 1-D Cosserat equation (directed curve) has been used which can be considered as simplified form of Navier-Stokes equations. Results are comparable with classical Rayleigh mode of circular jet when the aspect ratio (ratio of major to minor axis) is one. Growth rate of instability on an elliptic liquid jet under various conditions has been compared with those of a circular jet. Results show that in comparison with a circular jet, the elliptic jet is more unstable and by increasing the aspect ratio the instability grows faster. In addition, similar to the circular case, the effect of viscosity is diminishing the growth rate for the elliptic jet.

INTRODUCTION

Sprays are important in many practical applications such as combustion of liquid fuels, ink-jet printing, coatings, painting and agriculture. The dispersion of spray drops in a gas is important in order to bring about efficient heat and mass transfer between liquid and gas phases.

The breakup of a liquid jet emanating into another fluid has been quantitatively studied for more than a century. Plateau (1873) observed that the jet tends to break into segments that the spherical drop formed from each segment gives the minimum surface energy. Rayleigh (1879) showed that the jet breakup is the consequence of hydrodynamic instability. Neglecting the viscosity of the jet liquid, the ambient fluid, and gravity, he demonstrated that a circular cylindrical liquid jet is unstable with respect to disturbances of wavelengths larger than the jet circumference. Linear temporal analyses which have been established by Rayleigh remain as the fundamental bases of liquid jet instability studies. Weber (1931) considered the effects of the liquid viscosity as well as the density of the ambient fluid. Chandrasekhar (1961) took into account the liquid viscosity and the liquid density, and showed mathematically that the viscosity tends to reduce the breakup rate and increase the drop size.

Three-dimensional equations describing the nonlinear motion of free-surface flow are hopelessly complicated if analytical solutions are to be obtained. Therefore generating simplified equations that still capture the essential nonlinear physics of the problem is interesting. This is possible in situations where the fluid thread is long and thin, so that the fluid flow is directed mostly along the axis and the velocity field is effectively one-dimensional. In particular, this so-called slenderness assumption turns out to be generically valid close

to breakup and becomes exact asymptotically close to pinch-off (Eggers, 1994).

Green and Naghdi (1976) used the basic theory of a one-dimensional Cosserat continuum and presented nonlinear one-dimensional theory for a straight, circular, viscous jet. Rather than start with a three-dimensional theory based on a three-dimensional continuum model, one starts at the outset with a one-dimensional continuum model, called a Cosserat continuum that has structure provided by deformable vectors, called "directors", which are kinematically independent of the displacement and its derivatives. Then the principles of continuum physics such as invariance requirements and constitutive assumptions are employed to derive the one-dimensional equations and appropriate boundary conditions. Caulk and Naghdi (1979), using the Cosserat theory of Green and Naghdi (1976), considered the more general temporal stability of a noncircular viscous jet that also included the effects of a rigid rotation. Using incompressibility to eliminate the director velocity, and after eliminating the stress quantities, Bogy (1978a) listed the axisymmetric Cosserat jet equations. The Cosserat equations were used in a series of papers (Bogy 1978a, 1978b, 1979a, 1979b) focusing on linear stability.

Few studies have considered noncircular orifices, though there has been interest in them since the nineteenth century. In many instances when a fluid exits an elliptical orifice, the free jet behavior is oscillatory; from elliptical the cross-section becomes circular down the jet, then again elliptical but with major axis perpendicular to that of the elliptical orifice, then circular, then elliptical with major axis in the original direction and so on. Once the shape of the jet cross-section becomes circular, inertia continues to change the shape of the cross-section towards an ellipse. Oscillations occur due to the competing surface tension and inertial force. The single transformation behavior occurs in the absence of the surface tension. The axis-switching phenomenon of an elliptical jet can be used for the measurement of dynamic surface tension. Rayleigh developed a model to calculate the dynamic surface tension from the wavelength of axis-switching of an elliptical jet. Rayleigh's model was based on the assumption that the departure from circularity of the elliptic cross section of the jet is small, and his model did not incorporate the viscosity of the liquid. This was further improved by Bohr (1909) by including the viscosity into the model. Geer and Strikwerda (1983) developed asymptotic models and approximations for slender free surface jets. Their work begins with the three-dimensional equations for potential flow in the context of an inviscid one-dimensional director theory. They numerically studied the slender nonaxisymmetric equations. However, the issue of drop formation was not addressed.

In the context of breakup of elliptical liquid jets, the experiments carried out by Hoyt and Taylor (1962) show that it is not possible to identify any kind of regular breakup process as observed in the case of circular orifice liquid jets. Ho and Gutmark (1987) have shown that with an elliptical nozzle, the mass entrainment was three to eight times higher than an axisymmetric jet in combustion process. Kasyap *et al.* (2009) presented experimental results on the breakup of liquid jets issuing from elliptical orifices. They described the visual observations on elliptical jets by the characterization of the axis-switching process and described the breakup curves of elliptical and circular jets. They found that a liquid jet emanating from an elliptical orifice exhibits more unstable behavior or a faster breakup process than a corresponding circular liquid jet in a specific range of flow were axis switching was observed. Furthermore they found that increasing the orifice aspect ratio of the elliptical orifice in some ranges makes the elliptical liquid jet more unstable.

Although elliptical orifices have been used for practical applications such as liquid propellant rocket injectors, spray painting, pesticide spraying, and irrigation devices, a systematic study on the breakup behavior of liquid jets emanating from elliptical orifices has not yet been reported. The objective of this research is to investigate instability of liquid jets issuing from elliptical nozzles in Rayleigh mode and comparing its behavior with circular jets. Regarding complexity of equations for asymmetric geometry, 3-D Navier-Stokes will be simplified to derive 1-D equations. Linear solution will be performed by perturbation methods considering viscous and inviscid case.

NOMENCLATURE

A	wave amplitude
F	free surface of ellipse
I	identity tensor
Oh	Ohnesorge number
T	stress tensor
V	velocity
e	ratio of minor to major axis
k	wave number
n	unit outward normal to the interface
p	pressure
p_a	ambient pressure
t_a	boundary stress vector in ambient
t_f	boundary stress vector in jet
α	growth rate
β	dimensionless growth rate
κ	mean curvature
λ	wavelength
μ	dynamic viscosity of liquid
ρ	liquid density
σ	surface tension
φ_1	semi-major axis
φ_2	semi-minor axis

JET STABILITY ANALYSIS

Jet stability theory considers a liquid jet issuing from an orifice into a stationary, incompressible gas. The stability of liquid surface to perturbations is examined and ultimately leads to a dispersion equation that relates the growth rate, α, of an initial perturbation of infinitesimal amplitude to its wavelength,

λ. The governing equations for mass and momentum conservation are solved subject to boundary conditions at the interface that include a kinematic jump condition, a tangential stress balance, and a normal stress balance. The normal stress balance accounts for surface tension, dynamic pressure (inertia), and viscous (normal) force. In the tangential stress balance the gas is typically assumed to be inviscid. Solution of the dispersion equation shows predicted wave growth rate versus wave number as a function of Ohnesorge number.

The maximum wave growth rate, α_{max}, and the corresponding wavelength, λ_{max}, characterize the fastest growing (or most probable) waves on the liquid surface that are eventually responsible for the breakup. Any arbitrary form of disturbance can be constructed by superposition of all Fourier components. Each Fourier component has the form $Aexp(ikz+\alpha t)$ where A is the wave amplitude, $k=k_r+ik_i$, is the complex wave number whose real and imaginary parts give, respectively, the number of waves over a distance 2π and the exponential spatial growth rate per unit distance in the axial z-direction, and $\alpha=\alpha_r+i\alpha_i$ is the complex wave frequency, the real and imaginary part of which give, the exponential temporal growth rate and the frequency of the Fourier wave, respectively. (Lin and Reitz, 1998)

One-dimensional approximation

Bechtel *et al*. (1988) developed a one-dimensional model for three-dimensional viscoelastic free jets based on the assumption that the jet is slender. They started from the Navier-Stokes equation with assuming lowest order velocity field. Equations of motion are then derived, which depend only on the axial variable. They found that allowing the velocity profile to be an infinite power series in the x and y directions (figure 1) and applying the slender jet asymptotic, the same lowest order equations will be obtained. The leading-order versions of this procedure lead to a set of equations well known as the Cosserat equations. Their work is summarized as follows.

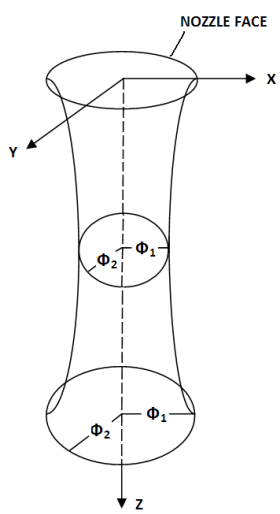

Fig.1. Liquid jet with elliptic cross section

Navier-Stokes equation is written as:

$$\rho(\frac{\partial V}{\partial t}+(V.\nabla)V)=\rho g+divT \tag{1}$$

where $T=-pI+T'$. Here V is velocity, T is stress tensor, T' is determinant part of the stress tensor, p is constraint pressure, ρ is the mass density (assumed constant) and ρg is gravitational body force (ignored here due to the rather high velocities).

The following assumptions were made; (a) cross section remains elliptical, (b) aerodynamic force from gas side is negligible, and (c) the velocity profile is assumed according to equation

$$V=[x\zeta_1(z,t)]e_1+[y\zeta_2(z,t)]e_2+[v(z,t)]e_3 \tag{2}$$

Where ζ_1 and ζ_2 are unknown functions and v is velocity component in z direction. Continuity equation gives,

$$\zeta_1+\zeta_2+v_z=0 \tag{3}$$

The first boundary condition is convection of free surface described below.

$$(\frac{\partial}{\partial t}+V.\nabla)F=0 \tag{4}$$

where F is free surface of ellipse

$$F(x,y,z,t)=\frac{x^2}{\phi_1^2(z,t)}+\frac{y^2}{\phi_2^2(z,t)}-1=0 \tag{5}$$

and ϕ_1 and ϕ_2 are semi-major and semi-minor axes respectively. From (4) and (5) following equations are derived.

$$\phi_{1t}+v\phi_{1z}-\phi_1\zeta_1=0\ ,\ \phi_{2t}+v\phi_{2z}-\phi_2\zeta_2=0 \tag{6}$$

The second boundary condition is continuity of normal stress at interface described in equation (7).

$$t_f-t_a=-\sigma\kappa n \tag{7}$$

Where t_f and t_a are the boundary stress vectors in the jet and ambient material, respectively. n is the unit outward normal to the interface and κ is mean curvature. In addition the ambient has been assumed to exert a constant pressure p_a, so $t_a=-p_a n$. Integrating momentum equations in the three directions on cross section, using Leibniz' rule for differentiation of integrals and divergence theorem, results in equation (8) for the three directions.

$$\frac{1}{4}\pi\rho\phi_1^3\phi_2(\zeta_{1t}+v\zeta_{1z}+\zeta_1^2)+2\mu\pi\phi_1\phi_2\zeta_1=p$$
$$+\phi_1\phi_2h(\phi_1,\phi_2)+\frac{1}{4}\mu\pi(\phi_1^3\phi_2\zeta_{1z})_z$$

$$\frac{1}{4}\pi\rho\phi_2^3\phi_1(\zeta_{2t}+v\zeta_{2z}+\zeta_2^2)+2\mu\pi\phi_1\phi_2\zeta_2=p$$
$$+\phi_1\phi_2h(\phi_2,\phi_1)+\frac{1}{4}\mu\pi(\phi_2^3\phi_1\zeta_{2z})_z \tag{8}$$

$$\pi\rho\phi_1\phi_2(v_t+vv_z)=-p_z-\phi_2\phi_{1z}h(\phi_1,\phi_2)$$
$$-\phi_1\phi_{2z}h(\phi_2,\phi_1)+2\mu\pi(\phi_1\phi_2v_z)_z$$

It is important to note that compared to the original Bechtel's equation the viscous terms in equation (8) are modified according to Caulk and Naghdi (1979). This modification is mainly due to the fact that the perturbation method used in this work is similar to that of Caulk and Naghdi's (1979).

Linear solution

As the perturbation is small, the products of its magnitude can be neglected, resulting in a linear analysis. Furthermore, temporal analyses consider the growth of the perturbation magnitude with time rather than along the jet axis, as is considered in spatial analyses. Linear expansion will result in equation (9).

$$v=v_0+\varepsilon v_0^{(0)},\qquad \phi_1=a+\varepsilon\phi_1^{(0)},\qquad \phi_2=b+\varepsilon\phi_2^{(0)}$$
$$p=p_0+\varepsilon p_0^{(0)} \tag{9}$$

where a and b are initial semi-major and semi-minor axes respectively. Pressure and velocity terms will be eliminated among equation (3), (6), and (8). With a selected frame of reference one may write v_0=0. For simplicity $\Phi_1^{(0)}$ has been written as Φ_1 and $\Phi_2^{(0)}$ as $\Phi_2^{(0)}$. Coefficients have been listed in Annex. Linearization of terms in (8) is as follows.

$$[(-\frac{1}{4}\pi\rho a^2b)\phi_{1ttzz}+(\pi\rho b)\phi_{1tt}+(-4\mu\pi b)\phi_{1tzz}+$$
$$(\frac{1}{4}\mu\pi a^2b)\phi_{1tzzzz}+(N_5)\phi_{1zzzz}+(N_3-bk_1)\phi_{1zz}]$$
$$+\left[(\pi\rho a)\phi_{2tt}+(-2\mu\pi b)\phi_{2tzz}+(N_2-al_1)\phi_{2zz}+(N_4)\phi_{2zzzz}\right]=0 \tag{10}$$

and

$$\left[(\pi\rho b)\phi_{1tt}+(-2\mu\pi b)\phi_{1tzz}+(M_2-bk_1)\phi_{1zz}+(M_4)\phi_{1zzzz}\right]+$$
$$[(-\frac{1}{4}\pi\rho b^2a)\phi_{2ttzz}+(\pi\rho a)\phi_{2tt}+(-4\mu\pi a)\phi_{2tzz}$$
$$+(\frac{1}{4}\mu\pi b^2a)\phi_{2tzzzz}+(M_5)\phi_{2zzzz}+(M_3-al_1)\phi_{2zz}]=0 \tag{11}$$

Additionally, surface tension term in (8) is linearized as,

$$h(\phi_1,\phi_2)=k_1+k_2\phi_1+k_3\phi_2+k_4\phi_{1zz}+k_5\phi_{2zz}$$
$$h(\phi_2,\phi_1)=l_1+l_2\phi_2+l_3\phi_1+l_4\phi_{2zz}+l_5\phi_{1zz} \tag{12}$$

Replacing following terms

$$\phi_1^{(0)}(z,t)=ae^{i(\alpha_1t-k_1z)},\phi_2^{(0)}(z,t)=be^{i(\alpha_2t-k_2z)} \tag{13}$$

in equations (10) and (11) results

$$(\frac{1}{4}\pi\rho a^2bk^2+2\pi\rho b)(i\alpha_1)^2+(6\mu\pi bk^2+\frac{1}{4}\mu\pi a^2bk^4)(i\alpha_1)$$
$$+(N_5+M_4)k^4-(N_3+M_2-2bk_1)k^2=0 \tag{14}$$

and

$$(\frac{1}{4}\pi\rho b^2ak^2+2\pi\rho a)(i\alpha_2)^2+(6\mu\pi ak^2+\frac{1}{4}\mu\pi b^2ak^4)(i\alpha_2)$$
$$+(N_4+M_5)k^4-(M_3+N_2-2al_1)k^2=0 \tag{15}$$

RESULTS AND DISCUSSION

By solving quadratic equations (14) and (15) which belong to the major and minor axes, respectively, the following dispersion equation will be derived,

$$i\alpha\sqrt{\frac{\rho R^3}{\sigma}}=-\frac{\sqrt{2}}{4}M+\sqrt{\frac{M^2}{8}+\frac{N}{2}} \tag{16}$$

where for the major axis,

$$M=\frac{e^{-0.25}Oh(3\sqrt{2}e+\frac{\sqrt{2}}{8}k^2R^2)k^2R^2}{e+\frac{1}{8}k^2R^2} \tag{17}$$

and

$$N=(\frac{\frac{e(3E_2-2F_1+3F_3)}{E_4+F_5}-k^2R^2}{e+\frac{1}{8}k^2R^2})(\frac{e\sqrt{e}(E_4+F_5)}{\pi}k^2R^2) \tag{18}$$

and for the minor axis,

$$M = \frac{e^{-0.25} Oh(3\sqrt{2} + \frac{\sqrt{2}}{8} ek^2R^2)k^2R^2}{1 + \frac{1}{8} ek^2R^2} \tag{19}$$

and

$$N = \left(\frac{\frac{e(3e^2F_2 - 2E_1 + 3e^2E_3)}{E_5 + F_4} - k^2R^2}{1 + \frac{1}{8} ek^2R^2}\right)\left(\frac{\sqrt{e}(E_5 + F_4)}{\pi} k^2R^2\right) \tag{20}$$

These equations have been plotted in figures 2 to 5. R is the radius of a jet which has the same cross sectional area of an ellipse, and e is the ratio of minor axis to major axis (i.e. inverse of aspect ratio). Ohnesorge number, Oh, and dimensionless growth rate, β, are defined as follows,

$$Oh = \frac{\mu}{\sqrt{\rho\sigma R}} \quad , \quad \beta = \alpha\sqrt{\frac{\rho R^3}{\sigma}} \tag{21}$$

Based on dispersion equation, results have been plotted for elliptic jets with different aspect ratios. Figure 2 shows that the disturbance growth rate in the elliptical case is larger and smaller than circular case for major and minor axis, respectively. By increasing the aspect ratio this difference becomes greater. Since the maximum growth rate is responsible for breakup, one can conclude that breakup length of elliptic jet is shorter than that of the circular jet which confirms the results obtained experimentally by Kasyap *et al.* (2009). In addition, these results show that in comparison with a circular jet, the elliptic jet is more unstable and by increasing the aspect ratio, the instability grows faster. Furthermore, as can be seen from figure 2, the range of unstable wave numbers in elliptic jet is larger than circular jet.

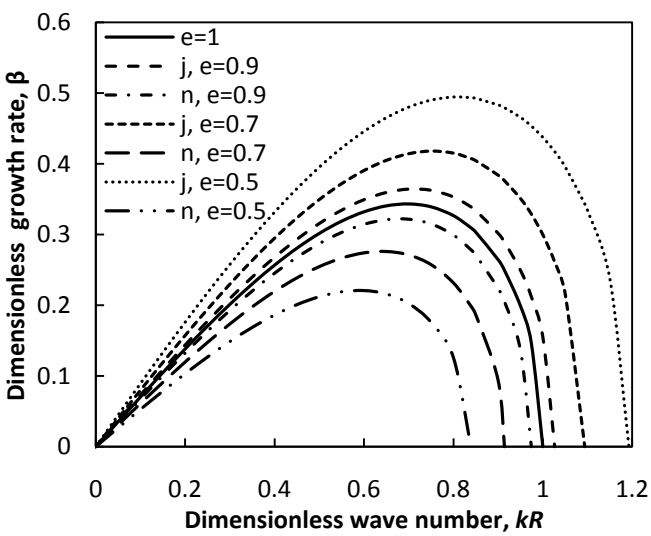

Fig.2. Growth rate β versus wavenumber kR for different aspect ratios in inviscid case (j: major axis, n: minor axis).

Figure 3 which had been presented by Caulk and Naghdi (1979) shows the effect of viscosity on the growth rate of disturbances on a circular jet. Figures 4 and 5 show the same effects on the major and minor axes of an elliptical jet. As can be seen, for both elliptical and circular jets the effect of viscosity is diminishing the growth rate and shifting the maximum growth rate to longer wavelengths. These results are comparable with those of Rayleigh (1879) and Weber (1931). Figure 3 shows that in the inviscid case, the difference between the results of one-dimensional model and Rayleigh's axisymmetric analysis is almost negligible (less than 0.25 %) which can serve as the validation for the 1-D modeling.

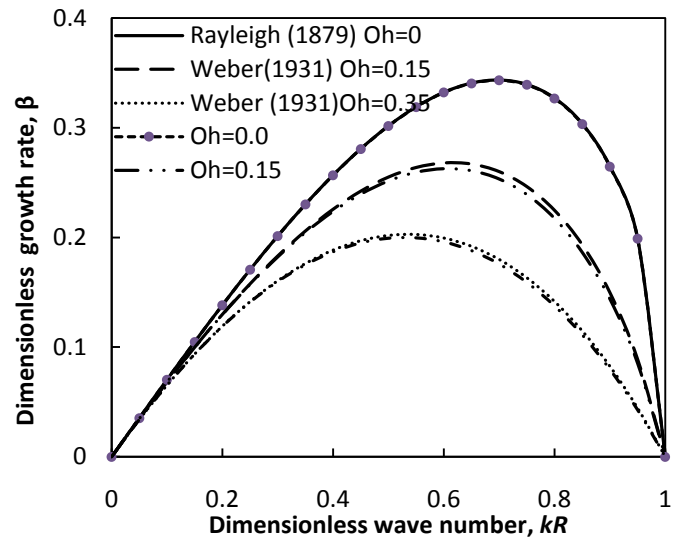

Fig.3. Comparison of the growth rate obtained by 1-D model with Rayleigh (1879) and Weber (1931) in circular case for different Oh numbers.

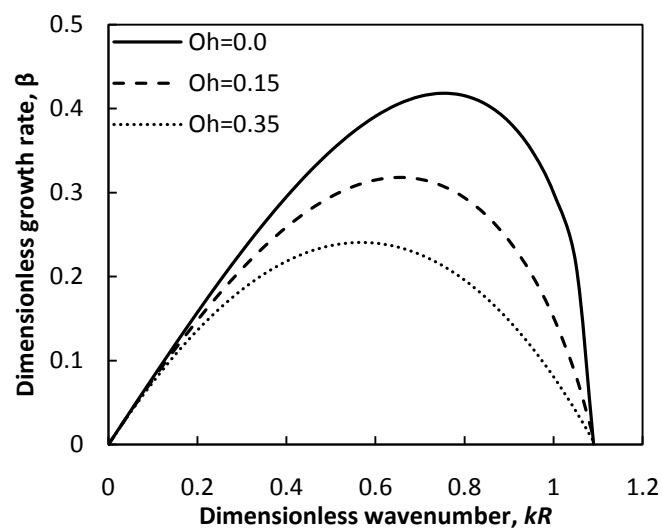

Fig.4. Growth rate of major axis for different Oh numbers for an ellipse with e=0.7

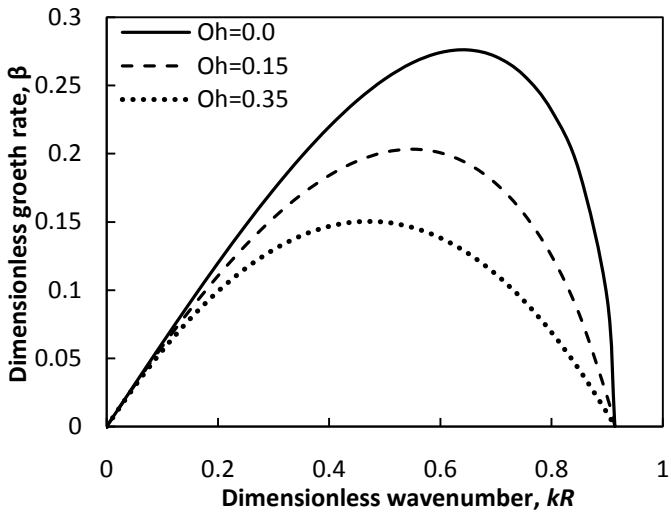

Fig.5. Growth rate of minor axis for different *Oh* numbers for an ellipse with *e*=0.7

CONCLUSION

Directed curve one-dimensional equation for liquid jet with elliptic cross section has been solved linearly with perturbation methods. The associated dispersion equation has been derived. Results show that in comparison with a circular jet, the elliptic jet is more unstable and by increasing the aspect ratio the instability grows faster and the range of unstable wave numbers is increased. Similar to the circular case the effect of viscosity is diminishing the growth rate and shifting the maximum growth rate to longer wavelengths.

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ANNEX

$$h(\phi_1,\phi_2)=\int_0^{2\pi} q\cos^2\theta d\theta$$

$$q=\sigma\{[(\phi_1\phi_{2z}\sin^2\theta+\phi_2\phi_{1z}\cos^2\theta)]^2+\phi_1^2\sin^2\theta+\phi_2^2\cos^2\theta\}^{-1.5}$$
$$\{(\phi_1^2\sin^2\theta+\phi_2^2\cos^2\theta)(\phi_{1zz}\phi_2\cos^2\theta+\phi_{2zz}\phi_1\sin^2\theta)$$
$$-2[(\phi_1\phi_{2z}-\phi_2\phi_{1z})\sin\theta\cos\theta][(\phi_2\phi_{2z}-\phi_1\phi_{1z})\sin\theta\cos\theta]$$
$$-\phi_1\phi_2[(\phi_{1z}^2\cos^2\theta+\phi_{2z}^2\sin^2\theta+1)]\}$$

$$E_1=\int_0^{2\pi}\frac{\cos^2\theta d\theta}{(\sin^2\theta+e^2\cos^2\theta)^{1.5}} \qquad F_1=\int_0^{2\pi}\frac{\cos^2\theta d\theta}{(e^2\sin^2\theta+\cos^2\theta)^{1.5}}$$

$$E_2=\int_0^{2\pi}\frac{\sin^2\theta\cos^2\theta d\theta}{(\sin^2\theta+e^2\cos^2\theta)^{2.5}} \qquad F_2=\int_0^{2\pi}\frac{\sin^2\theta\cos^2\theta d\theta}{(e^2\sin^2\theta+\cos^2\theta)^{2.5}}$$

$$E_3=\int_0^{2\pi}\frac{\cos^4\theta d\theta}{(\sin^2\theta+e^2\cos^2\theta)^{2.5}} \qquad F_3=\int_0^{2\pi}\frac{\cos^4\theta d\theta}{(e^2\sin^2\theta+\cos^2\theta)^{2.5}}$$

$$E_4=\int_0^{2\pi}\frac{\cos^4\theta d\theta}{(\sin^2\theta+e^2\cos^2\theta)^{0.5}} \qquad F_4=\int_0^{2\pi}\frac{\cos^4\theta d\theta}{(e^2\sin^2\theta+\cos^2\theta)^{0.5}}$$

$$E_5=\int_0^{2\pi}\frac{\sin^2\theta\cos^2\theta d\theta}{(\sin^2\theta+e^2\cos^2\theta)^{0.5}} \qquad F_5=\int_0^{2\pi}\frac{\sin^2\theta\cos^2\theta d\theta}{(e^2\sin^2\theta+\cos^2\theta)^{0.5}}$$

$$k_1=-\sigma abE_1 \qquad M_1=abl_1$$
$$k_2=-\sigma bE_1+3\sigma a^2bE_2 \qquad M_2=abl_3+bl_1$$
$$k_3=-\sigma aE_1+3\sigma ab^2E_3 \qquad M_3=abl_2+al_1$$
$$k_4=\sigma bE_4 \qquad M_4=abl_5$$
$$k_5=\sigma aE_5 \qquad M_5=abl_4$$

$$l_1=-\sigma abF_1 \qquad N_1=abk_1$$
$$l_2=-\sigma aF_1+3\sigma b^2aF_2 \qquad N_2=abk_3+ak_1$$
$$l_3=-\sigma bF_1+3\sigma ba^2F_3 \qquad N_3=abk_2+bk_1$$
$$l_4=\sigma aF_4 \qquad N_4=abk_5$$
$$l_5=\sigma bF_5 \qquad N_5=abk_4$$