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OPTIMAL GROWTH AND TRANSITION TO TURBULENCE IN MAGNETOHYDRODYNAMIC DUCT FLOW

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ABSTRACT

We consider the flow of an electrically conducting fluid in a duct in the presence of a constant magnetic field perpendicular to the flow. The technologically relevant approximation of small magnetic Reynolds number is adopted. The focus of investigation is on the nonlinear mechanism of transition consisting of transient growth and subsequent breakdown of finite amplitude perturbations. Numerical analysis demonstrates that the strongest growth is experienced by perturbations localized in the sidewall boundary layers parallel to the imposed magnetic field. This result and the direct numerical simulations of the transition process indicate that the commonly accepted picture of the transition in MHD duct based on the numerical and theoretical analysis of the flow in the Hartmann channel is misleading. The flow may become turbulent within the sidewall layers long before the Hartmann layers on the walls perpendicular to the magnetic field are able to sustain nonlinear transition.

INTRODUCTION

Constant magnetic fields are applied in several metallurgical and materials processing technologies, for example, in continuous casting of steel [1] and growth of large semiconductor crystals [2]. The purpose is usually to control the melt flow by suppressing unwanted fluid motions. The magnetic fields are strong (for example, fields up to 1 Tesla are not uncommon in the continuous steel casting), which results in strong suppression of the flow so that it acquires laminar, transitional, or weakly turbulent state even though the Reynolds number is typically large.

The technological flows of liquid metals are usually characterized by small values of the magnetic Reynolds number $Re_m = UL/\eta$, where U and L are the typical velocity and length scale and $\eta = (\sigma\mu_0)^{-1}$ is the magnetic diffusivity, σ and μ_0 being the electric conductivity of the fluid and magnetic permeability of vacuum, respectively. The distinctive feature of the low-Re_m flows is the one-way coupling between the magnetic field and the flow. The flow is affected by the Lorentz force. At the same time, the magnetic field perturbations associated with induced currents are negligible by comparison with the imposed field. This separates the liquid metal MHD from its high-Re_m counterpart in plasma and geo- and astrophysical applications.

In this paper, we analyze the transition to turbulence in the flow of an electrically conducting incompressible fluid in a rectangular duct with electrically insulating walls. A steady uniform magnetic field is imposed in the direction parallel to one set of wall and perpendicular to the flow. This flow is an archetypal flow of liquid metal magnetohydrodynamics, since it represents the key features found in the MHD applications: suppression and anisotropy, uniform core flow, interaction between the magnetic field and mean shear, and typical MHD boundary layers.

Remarkably, very little is known about the instability of the MHD duct flow. The linear stability problem has not been solved. It can be noted that such a solution would predict the critical Reynolds numbers much larger than observed in reality and, thus, would fail to provide an explanation for the transition. The reason is that, as illustrated in figure 1, the MHD duct flow belongs to the same type of wall-bounded parallel flows as the classical hydrodynamic flows in pipes and channels. Base flow profile is characterized by monotonous variation of velocity from the maximum at the center of the duct to zero at the walls. Such flows are known to experience transition to turbulence at much lower Reynolds numbers than predicted by the analysis of infinitesimal exponentially growing or decaying perturbations.

The accepted explanation of the inability of the linear stability theory to predict the transition in wall-bounded parallel flows is the concept of bypass transition [3]. Due to the non-normality of the linearized operator, a combination of eigenfunctions of a linear problem can experience significant transient growth prior to eventual exponential decay. An essential physical mechanisms of the growth is the 'lift-up effect', the redistribution of the mean flow velocity by the perturbations [4]. Considered in the framework of a nonlinear problem, the transient growth can produce finite-amplitude amplification of the perturbations sufficient to temporarily modify the base flow to the state unstable to secondary infinitesimal perturbations. A transition according to the bypass scenario was demonstrated, among other flows, for the plane channel [5], pipe [6]), and rectangular duct [7] flows. In all cases, the 'optimal' modes providing the strongest transient growth were found to have the form of streamwise-independent rolls evolving into streamwise streaks.

The effect of the imposed magnetic field on the transition scenario was studied in [8] and [9] for the channel flows, in which the magnetic field was, respectively, in the wall-normal (Hartmann channel) and spanwise directions. In the Hartmann case, the computations showed the optimal modes in the form of streamwise rolls evolving into streaks limited to the Hartmann boundary layers. At the reasonable initial amplitude of the optimal modes, transition was realized at the values of Reynolds number close to the critical number detected in the experiments [10].

In the case of spanwise magnetic field, the base flow remains the same as in the hydrodynamic channel flow. The imposed magnetic field has, however, strong impact on the nature of the optimal modes. As found in [9], the strongest amplification is provided by rolls oriented at an oblique angle to the flow direction. The rolls evolve into streak-like structures in the course of growth. The orientation angle increases with the strength of the magnetic field and eventually reaches the limit corresponding to purely spanwise rolls (TS-waves). Direct simulations of transition show that two symmetric oblique rolls with opposite values of the spanwise wavenumber can effectively trigger the transition serving as secondary perturbations to each other.

In this paper, we present the results of a numerical study of the bypass transition in the MHD duct flow. The study starts with identifying the optimal modes in a wide range of the strengths of the magnetic field and duct aspect ratios. This part is presented in great detail in our recent publication [11]. The analysis is continued by simulation of the entire transition.

MODEL AND PARAMETERS

We consider a flow of an incompressible electrically conducting fluid in a duct with electrically insulating walls at $z = \pm d/2$ and $y = \pm a/2$. The flow in the streamwise direction x is driven by applied pressure gradient, which is maintained to conserve the bulk flow rate. Uniform constant magnetic field $B_0 = B_0 e$ is imposed, where $e = (0,0,1)$. In the quasi-static approximation valid at low magnetic Reynolds number, the non-dimensional governing equations and boundary conditions are

$$\frac{\partial v}{\partial t} + (v\cdot\nabla)v = -\nabla p + \frac{1}{Re}\nabla^2 v + N\left(-\nabla\phi + (v\times e)\right)\times e, \quad (1)$$

$$\nabla\cdot v = 0, \quad (2)$$

$$\nabla^2\phi = \nabla\cdot(v\times e), \quad (3)$$

$$v_x = v_y = v_z = \frac{\partial\phi}{\partial z} = 0 \quad \text{at } z = \pm d/(2L), \quad (4)$$

$$v_x = v_y = v_z = \frac{\partial\phi}{\partial y} = 0 \quad \text{at } y = \pm a/(2L). \quad (5)$$

The flow is assumed periodic in the x-direction.

The problem has three independent non-dimensional parameters: the Reynolds number

$$Re \equiv \frac{U_0 L}{\nu}, \quad (6)$$

the aspect ratio

$$r \equiv \frac{a}{d}, \quad (7)$$

and the magnetic interaction parameter

$$N \equiv \frac{Ha^2}{Re}, \quad (8)$$

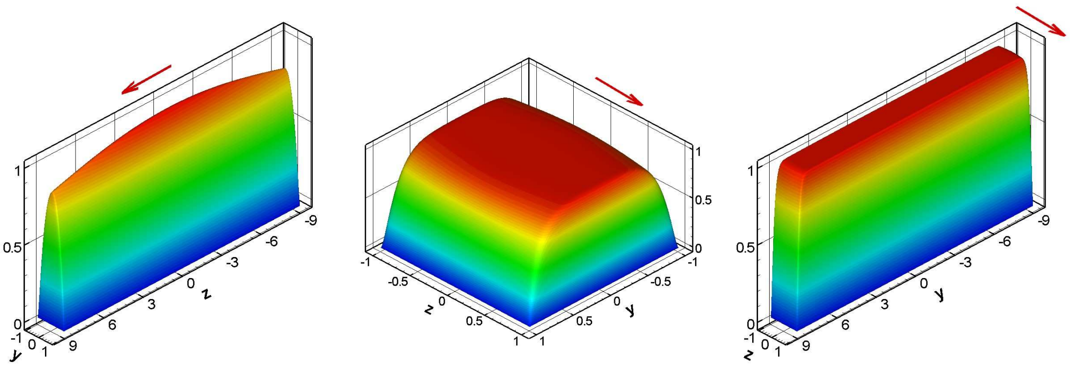

FIGURE 1. Base flow for $Ha = 50$, Reynolds number $Re = 5000$ and the aspect ratio r equal to 1/9 (left), 1 (middle), and 9 (right). Contour plots of streamwise velocity are shown. Arrows indicate the orientation of the magnetic field.

where Ha is the Hartmann number,

$$Ha \equiv LB_0\sqrt{\frac{\sigma}{\rho\nu}}. \tag{9}$$

As typical scales, we used the half of the smaller channel width $L \equiv \min(a/2, d/2)$ and the centerline velocity of the laminar base flow U_0. The scales of time, pressure magnetic field and electric potential are L/U_0, ρU_0^2, B_0, and LU_0B_0.

The optimal perturbations are found as Fourier modes of streamwise wavenumber α computed by the fixed-point iteration method based on the direct and adjoint equations [11]. The perturbations are optimized in the sense that modes having the maximum amplification are found as function of Re, Ha, r, and the time T, at which the amplification is measured. The global optimal modes presenting the maximum amplification over all times T are also found. The problem is solved numerically using the projection-type explicit finite-difference scheme of the second order. The grid is orthogonal and non-uniform with clustering applied near the wall in order to provide the necessary resolution of the MHD boundary layers. The grid includes up to 82 points in the direction of the shorter side and up to 288 points in the longer side direction. The time integration is based on the Adams-Bashfort scheme of the second order. Further details of the numerical procedure and verification data are available in [11]).

The Reynolds number is kept at 5000. The Hartmann number varies between 0 and 50. A wide range of aspect ratios is investigated with r taking values 1/9, 1/3, 1, 3, and 9. Importance of considering various aspect ratios stems from the possible similarity between the duct flow and the MHD channel flows with spanwise and wall-normal magnetic field, which correspond, respectively, to the asymptotic limit of duct with $r \to 0$ and $r \to \infty$. To stress the possible similarity, the flows in ducts with $r < 1$ are in the following referred to as the 'spanwise case' and the flows in ducts with $r > 1$ as the 'Hartmann case'.

OPTIMAL MODES

The base flow is illustrated in figure 1. One can see the characteristic features of the MHD flow: a nearly uniform core and specific boundary layers: Hartmann layers of thickness Ha^{-1} at the walls normal to the magnetic field and sidewall (Shercliff) layers of thickness $Ha^{-1/2}$ at the walls parallel to the magnetic field.

Before presenting the results for the optimal modes, it is useful to recall the general features of the effect of the imposed magnetic field on the flow structure in the case of low magnetic Reynolds number. As derived from the equations (see, e.g., [12]) and demonstrated in numerical computations of MHD turbulence (see, e.g., [13]), the effect is two-fold. The Joule dissipation of induced electric current leads to suppression of the flow. Furthermore, since the Joule dissipation is proportional to the cosine squared of the angle between the magnetic field lines and the wavenumber vector of the velocity, the suppression is the strongest for the flow modes with strong variation in the magnetic field direction and zero for the modes uniform in this direction. Applied to the case of growing perturbations, this means that the growth is suppressed and that stronger growth tends to be experienced by perturbations aligned with the magnetic field. This effect was manifested by oblique character of the optimal modes in the channel with spanwise magnetic field [9].

In the hydrodynamic case we found, in agreement with [7], that, except for small times T, the maximum amplification is due to the streamwise rolls evolving into streamwise streaks.

The global amplification data for non-zero magnetic fields are summarized in figure 2. In addition to the optimal modes obtained for modes with arbitrary streamwise wavenumber, the maximum amplifications for purely streamwise modes with $\alpha = 0$ are included. One can clearly see the effects of the imposed magnetic field. One effect is the suppression of perturbations by the magnetic field. The amplification reduces with growing Hartmann number. Another effect is that the strongest growth is consistently by non-streamwise perturbations.

The effect of the duct aspect ratio on the amplification is further illustrated in figure 3. The amplification factor maximized over wavenumbers α is shown as a function of time T for the case $Re = 5000$ and $Ha = 10$. The results of a periodic channel with either wall-normal or spanwise magnetic field are shown for comparison in dots.

One can see in the left plot of figure 3 that at $r \to 0$ the amplification factor G approaches the factor obtained in a channel with spanwise magnetic field [9]. The maximum amplification increases significantly when r is reduced to $r \ll 1$. This behavior appears natural since in this case perturbations localized near the long side walls parallel to the magnetic field are expected to dominate. It is not surprising that their behavior becomes similar to the behavior of the perturbations in the channel as $r \to 0$.

On the contrary, for large r (long Hartmann walls) the amplification factor G is significantly higher than for periodic Hartmann channel (right plot of figure 3). In fact, r has little effect on the amplification. This observation suggests that the Shercliff layers, which do not depend on r when $r \gg 1$, still provide the dominant contribution to the amplification even though they are much shorter than the Hartmann walls.

Our interpretation of the results presented in figure 3 is confirmed by visualization of the optimal modes shown in figures 4 and 5. The pattern, which is visualized by the streamwise velocity perturbations, corresponds to the global optimal modes at the maximum amplification time. It is clearly seen that the structures are largely confined to the Shercliff boundary layers near the sidewalls (walls parallel to the magnetic field). Remarkably, this is also true in the case $r > 1$. This clearly separates the duct flow from the Hartmann channel flow. Also important is that in all cases, the optimal perturbations are not streamwise ($\alpha \neq 0$).

DIRECT NUMERICAL SIMULATION OF TRANSITION

Further insight into the nature of the transition can be obtained from fully non-linear simulations. Systematic simulations are currently under way. First results are presented in figure 6. Transition in the flow at $Re = 5000$, $Ha = 10$, and $r = 6$ is triggered by random $3D$ noise imposed at the initially laminar flow state. The figure shows snapshots of the mean velocity profile corresponding to various phases of transition. It is clearly seen that the basic conclusion of the linear analysis is confirmed. The instability originates in the boundary layers near the side walls. The transition process proceeds with the perturbations growing in amplitude and spreading into the core flow. Similar behavior was observed at $r = 3$ and $r = 1$.

CONCLUSION

The main conclusion of our work is that the transition to turbulence in the MHD duct flow occurs first in the Shercliff layers near the walls parallel to the magnetic field. This is true in a wide range of aspect ratios, including, rather unexpectedly, the cases of a duct strongly elongated in the direction perpendicular to the magnetic field, in which the Shercliff layers occupy a very small fraction of the duct. The transition appears to follow the conventional scenario of transient growth of roll-like structures evolving into streaks and secondary instability and breakdown of the streaks. The scenario is modified by the magnetic field in the sense that the optimal modes causing the transition are not purely streamwise.

One implication of the results is that we have to be careful while interpreting the data of liquid metal experiments. Visual observation of flow structure is impossible in such experiments and the transition is typically detected by the change in the Reynolds number dependency of the friction coefficient (see, e.g. [10]). This event is usually attributed to the lost of stability by the Hartmann layers. This may be correct in the cases of very high Hartmann numbers (as indicated by the agreement between the results of [10] and our analysis of the Hartmann channel flow [8]). In such cases, the turbulence generated in the earlier transition within the Shercliff layers would likely remain confined to these layers and, thus, would not affect the friction. In flows with moderate vaules of Ha, however, the possibility that the observed sharp bend of the friction curve corresponds to the instability of Shercliff layers should be considered.

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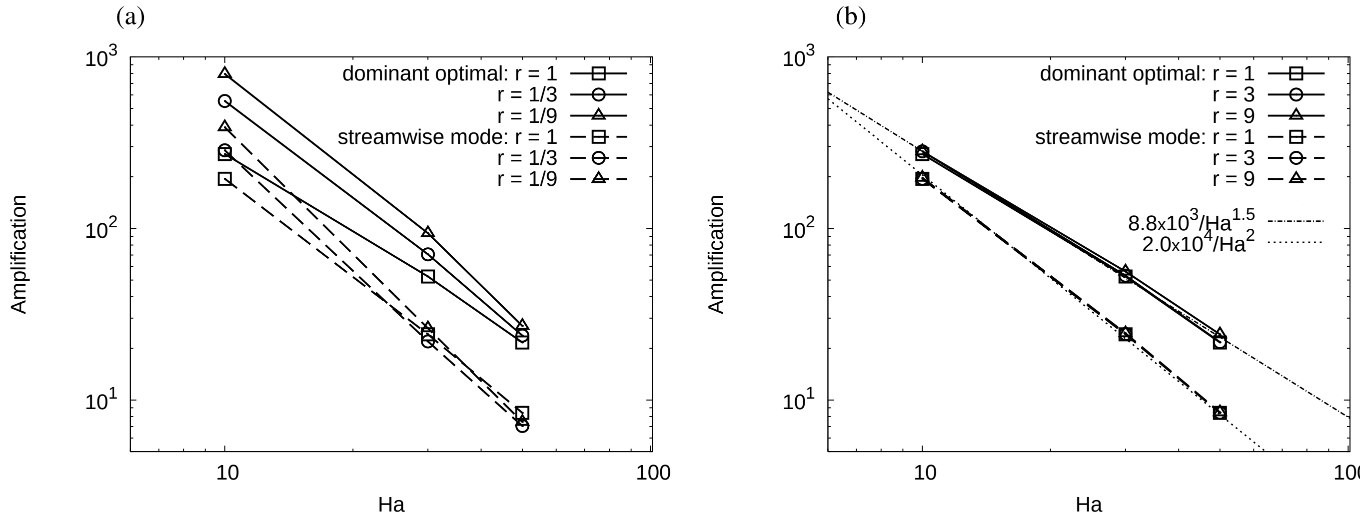

FIGURE 2. Global maximum amplification $M_{tot}(Re,Ha,r)$ (bold line) and maximum amplification of streamwise perturbations $M_{stream}(Re,Ha,r)$ (dashed line) are shown at $Re = 5000$ for different Hartmann numbers Ha and different aspect ratios r. The spanwise case $r \leq 1$ is presented in *(a)*, the Hartmann case $r \geq 1$ in *(b)*.

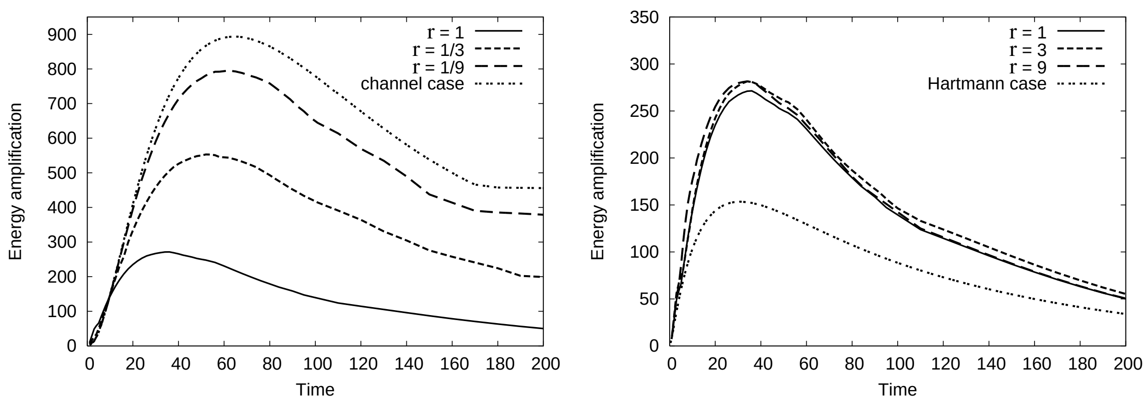

FIGURE 3. Energy amplification for $Re = 5000$ and $Ha = 10$: $r < 1$(left), $r > 1$(right).

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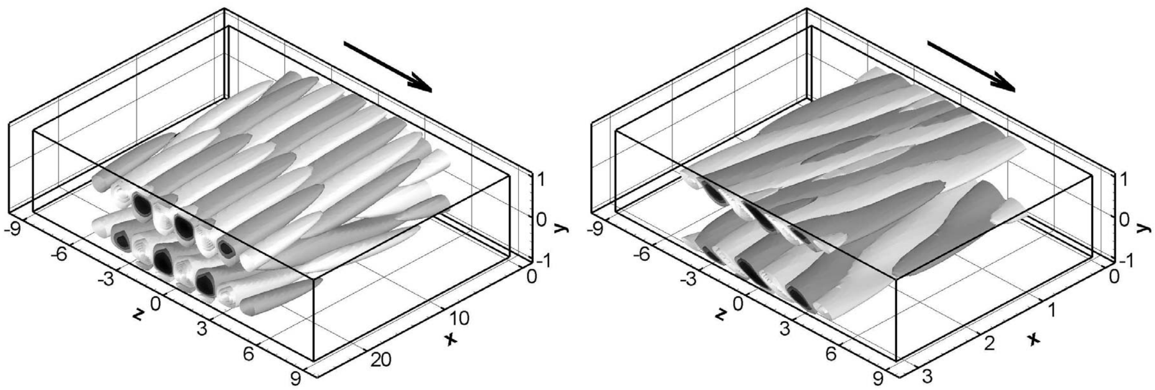

FIGURE 4. Spanwise case ($r = 1/9$). The optimal perturbation at T_{opt} corresponding to maximum amplification $M_{opt}(Re, Ha, r)$ is shown using iso-surfaces of the streamwise velocity perturbation for Reynolds number $Re = 5000$ and Hartmann numbers $Ha = 10$ (left), $Ha = 50$ (right). The arrow indicates the orientation of applied magnetic field B.

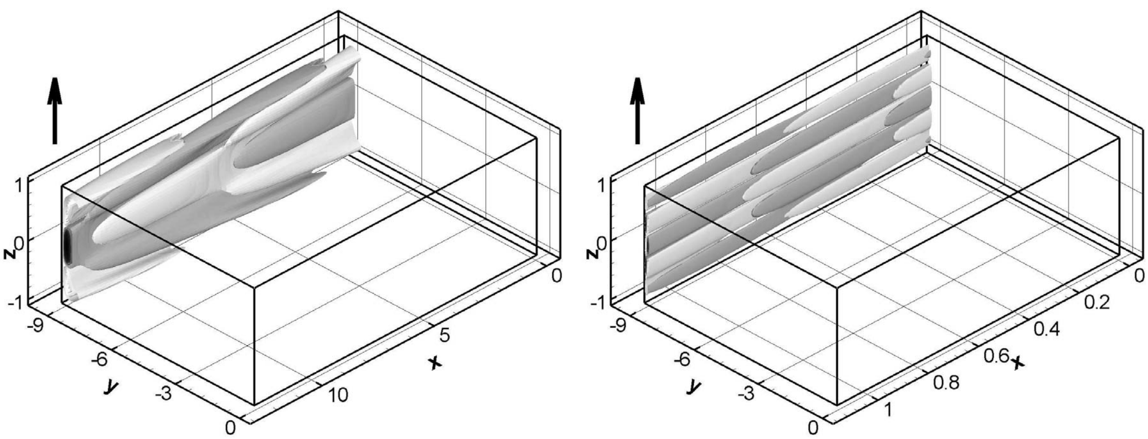

FIGURE 5. Hartmann case ($r = 9$). The optimal mode corresponding to the maximum amplification $M_{opt}(Re, Ha, r)$ for Reynolds number $Re = 5000$ and Hartmann numbers $Ha = 10$ (left), $Ha = 50$ (right). Spatial structure at T_{opt} is shown by iso-surfaces of the streamwise velocity in one half of the domain. The arrow indicates the orientation of applied magnetic field B.

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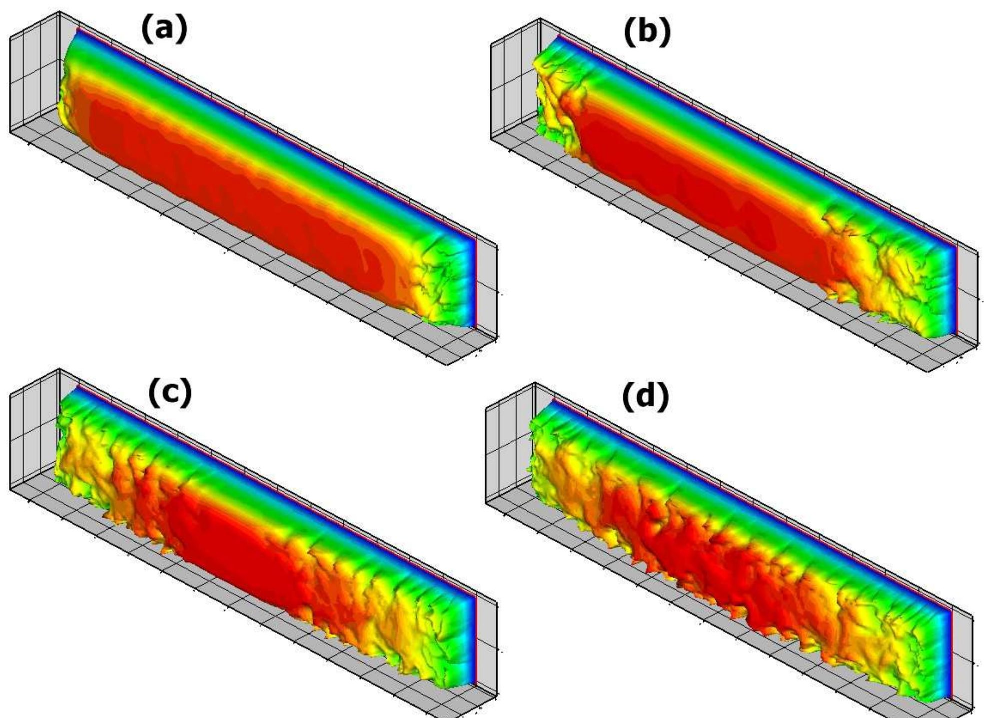

FIGURE 6. Streamwise velocity in the duct cross-section at different stages of transition for $Re = 5000$, $Ha = 10$, $r = 6$.