Error-estimation and adaptivity using operator-customized finite-element wavelets

R. Sudarshan[a], K. Amaratunga[a,*], T. Grätsch[b]

Departments of [a]Civil and Environmental Engineering and [b]Mechanical Engineering, Massachusetts Institute of Technology, 77 Massachusetts Avenue, Cambridge, MA 02139, USA

Abstract

We describe how wavelets constructed out of finite-element interpolation functions provide a convenient mechanism for both error-estimation and adaptivity in finite-element analysis. This is done by posing an adaptive refinement problem as one of compactly representing a signal (the solution to the governing partial differential equation (PDE) or boundary integral equation (BIE), with isolated features of interest. To compress the solution in an efficient manner, we first compute approximately the details to be added to the solution on a coarse mesh in order to obtain the solution on a finer mesh (the estimation step) and then compute exactly the coefficients corresponding to only those basis functions contributing significantly to the details (the adaptation step). In this sense, therefore, the proposed approach is unified, since the basis functions used for error-estimation are exactly the same as those used for adaptive refinement. We illustrate the application of the proposed techniques for goal-oriented adaptivity for second- and fourth-order linear, elliptic PDEs.

Keywords: Multiresolution analysis; Customized wavelets; Goal-oriented adaptivity

1. Introduction

Multiresolution signal-processing techniques such as wavelets and filter-banks are useful in generating compact, adaptive representations of data with localized features. They are therefore highly appropriate as tools for adaptive computational modeling. In constructing wavelets for this application, a particularly useful starting point is the hierarchical finite-element framework [1]. Using procedures such as stable completion [2], the elementary wavelets arising out of this framework may be customized such that the solution (and any linear functional of the solution) may be computed in an entirely incremental manner. Further, the wavelets are customized to give rise to well-conditioned stiffness matrices. This in turn permits the use of inexpensive iterative solvers to estimate the solution at each resolution.

In this article, we consider the problem of *goal-oriented* error estimation and adaptivity [3–5], where the mesh is adapted to minimize the distance between the true solution and the adapted finite-element solution in terms of output functionals of interest rather than the energy norm. In our approach, the estimate for the error in the quantity of interest is provided directly by considering the contribution of wavelets at each level. This is therefore unlike approaches such as the one proposed by Becker and Rannacher [3] (in which element-wise application of the Cauchy–Schwarz inequality leads to overestimation of the error) or that proposed by Prudhomme and Oden [4] (which involves the computation of lower and upper bounds in the energy norm of errors in both the solution and the influence function). In this article, we also provide a priori accuracy bounds of the error estimates at each level of resolution.

2. Theory and implementation

2.1. Problem definition

Consider a domain $\Omega \subset \mathbb{R}^n$, a Sobolev space $V(\Omega)$, a symmetric, continuous, and coercive bilinear form $a(\cdot,\cdot)$: $V \times V \rightarrow \mathbb{R}$, and a bounded linear form $l(\cdot) : V \rightarrow \mathbb{R}$.

* Corresponding author. Tel.: +1 617 258 0217; Fax: +1 617 253 6324; E-mail: kevina@mit.edu

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Therefore, there exists a unique solution, $u \in V$, to the primal problem:

$$a(u,v) = l(v) \quad \forall v \in V$$

Given a bounded linear functional $Q(\cdot)$: $V \rightarrow \mathbb{R}$ (the quantity of interest), the problem is to determine the quantity $Q(u)$ in an efficient and yet accurate manner. This is done by considering an error tolerance, ε, and constructing a computationally inexpensive approximation $\bar{u} \in V$, such that

$$d(u,\bar{u}) \triangleq |Q(u) - Q(\bar{u})| \leq \varepsilon$$

We also define the solution to the dual problem (called the *influence function*), $z \in V$, satisfying

$$a(z,v) = Q(v) \quad \forall v \in V \quad \Rightarrow \quad Q(u) = a(z,u) = l(z)$$

2.2. Multiresolution approximation

We now consider a multiresolution analysis of V consisting of:

1. A ladder of nested approximation spaces:

$$V_0 \subset V_1 \ldots \subset V_j \ldots \subset V \quad \text{such that} \quad \text{clos}\bigcup_{j=0}^{\infty} V_j = V$$

The approximation spaces are chosen as those generated by *complete* and *compatible* finite-element interpolation functions defined on nested discretizations of Ω.

2. Complementary (wavelet) spaces W_j, such that

$$V_{j+1} = V_j \oplus W_j \quad \text{and} \quad a(u,v) = 0 \quad \forall u \in V_j, v \in W_j \tag{1}$$

The complementary spaces are constructed by performing modified Gram–Schmidt orthogonalization of hierarchical basis functions, as described in Amaratunga and Sudarshan [6]. Let $\mathcal{M}(j)$ denote the index set of all basis functions (the 'wavelets') in W_j.

The solution of the primal and dual problems (respectively, u and z) can then be expanded as:

$$u = u_0 + \sum_{j=0}^{\infty} r_j \quad \text{and} \quad z = z_0 + \sum_{j=0}^{\infty} \eta_j$$

where u_0, $z_0 \in V_0$ are, respectively, the 'coarse' components of the solution and the influence function and r_j, $\eta_j \in W_j$ are the corresponding details that enable the transition from a coarser resolution to a finer resolution.

2.3. Adaptive refinement algorithm

We now summarize below the basic steps in our multiresolution approach for goal-oriented error estimation and adaptivity:

1. Given the solution at level j, u_j, estimate the details r_j. The details are computed by solving a system of equations of the form:

$$\mathbf{C}_{j+1} r_j = g_j$$

where $\mathbf{C}_{j+1} = a(w_j, w_j^{\mathsf{T}})$ is the stiffness matrix corresponding to the addition of scale-orthogonal wavelets at level j and $g_j = l(w_j)$ is the corresponding load vector. For wavelets constructed using Gram–Schmidt orthogonalization of the hierarchical basis, the stiffness matrix, $\mathbf{C}_{j+1}$ is the Schur's complement of the hierarchical basis stiffness matrix [6],

$$\mathbf{C}_{j+1} = \mathbf{C}_{j+1}^{\mathrm{HB}} - (\mathbf{B}_{j+1}^{\mathrm{HB}})^{\mathsf{T}} \mathbf{K}_j^{-1} \mathbf{B}_{j+1}^{\mathrm{HB}} \quad \text{and} \tag{2a}$$

$$g_j = g_j^{\mathrm{HB}} - (\mathbf{B}_{j+1}^{\mathrm{HB}})^{\mathsf{T}} u_j \tag{2b}$$

where K_j is the stiffness matrix at level j, $\mathbf{B}_{j+1}^{\mathrm{HB}}$ is the interaction matrix between the scaling functions and hierarchical basis functions at level j, and $\mathbf{C}_{j+1}^{\mathrm{HB}}$ are the stiffness matrices corresponding to the hierarchical basis functions alone at level j. Computing all the detail coefficients at level j by inverting $\mathbf{C}_{j+1}$ is extremely expensive, since it is normally a dense matrix. However, $\mathbf{C}_{j+1}$ and $\mathbf{C}_{j+1}^{\mathrm{HB}}$ (a sparse matrix) are equivalent in the sense that for any vector of detail coefficients, r_j,

$$(1-\gamma^2)\, r_j^{\mathsf{T}} \mathbf{C}_{j+1}^{\mathrm{HB}} r_j \leq r_j^{\mathsf{T}} \mathbf{C}_{j+1} r_j \leq r_j^{\mathsf{T}} \mathbf{C}_{j+1}^{\mathrm{HB}} r_j, \quad \gamma \in [0,1)$$

The matrix $\mathbf{C}_{j+1}^{\mathrm{HB}}$ therefore is a good approximation to the matrix $\mathbf{C}_{j+1}$. Hence, let

$$\tilde{r}_j = (\mathbf{C}_{j+1}^{\mathrm{HB}})^{-1} g_j = (\mathbf{C}_{j+1}^{\mathrm{HB}})^{-1} \left(g_j^{\mathrm{HB}} - (\mathbf{B}_{j+1}^{\mathrm{HB}})^T u_j \right)$$

be an estimate for r_j. It can be shown that the error in the quantity of interest in approximating r_j with $\tilde{r}_j$ can be bounded above as:

$$\left| Q\,(r_j - \tilde{r}_j) \right| = \left| a\,(\eta_j, r_j - \tilde{r}_j) \right| \leq \frac{\gamma^2}{\sqrt{1-\gamma^2}} \|r_j\|_{\mathrm{E}} \|\eta_j\|_{\mathrm{E}}$$

In order to compute the estimates $\tilde{r}_j$ more efficiently, we use the property that the condition number κ of $\mathbf{C}_{j+1}^{\mathrm{HB}}$ is bounded uniformly [1]. Hence we can approximately invert $\mathbf{C}_{j+1}^{\mathrm{HB}}$ using only a few conjugate gradient iterations with diagonal scaling. The error in estimating the functional of interest by approximating $\tilde{r}_j$ with $\tilde{r}_j^{(k)}$ obtained after k conjugate gradient iterations can be bounded above as

$$\left| Q\left(\tilde{r}_j - \tilde{r}_j^{(k)} \right) \right| \leq \frac{2}{\sqrt{1-\gamma^2}} \|\tilde{r}_j\|_{\mathrm{E}} \|\eta_j\|_{\mathrm{E}} \left(\frac{\sqrt{\kappa}-1}{\sqrt{\kappa}+1} \right)^k$$

2. Compute the contribution of each detail coefficient to the linear functional, i.e. $|Q\,(\tilde{r}_{j,m}^{\mathrm{T}} w_{j,m})|$, $m \in \mathcal{M}(j)$. The dual load vector $Q_{j,m} = Q(w_{j,m})$ is computed as in Eq. (2b).
3. Given a threshold τ_j, select the set $\mathcal{M}'(j)$ of wavelets to be retained, such that

$$\left|Q\left(\tilde{r}_{j,m'}^{\mathrm{T}} w_{j,m'}\right)\right| \geq \tau_j \max_{m \in \mathcal{M}(j)} \left|Q\left(\tilde{r}_{j,m}^{\mathrm{T}} w_{j,m}\right)\right| \quad \forall m' \in \mathcal{M}'(j)$$

   and solve for the retained details exactly by constructing only those scale-orthogonal wavelets corresponding to the set $\mathcal{M}'(j)$.
4. Compute the contribution of retained details to the influence function (needed for step 2 at level $j + 1$).
5. Finally, we can estimate the contribution of the discarded detail coefficients as

$$d(u_{j+1},\bar{u}_{j+1}) \approx \left|\sum_{m \in \mathcal{M}''(j)} Q\left(\tilde{r}_{j,m}^{\mathrm{T}} w_{j,m}\right)\right| \tag{3}$$

3. Numerical experiments

In this section, we present a few representative numerical results for goal-oriented adaptivity for fourth-order problems (bending deformation of thin plates).

3.1. L-shaped domain

We first consider the domain shown in Fig. 1a with the goal of determining accurately the displacements at point R. The thickness, Young's modulus, and Poisson's ratio are taken, respectively, as $h = 1$, $\mathrm{E} = 1.0 \times 10^7$, and $\nu = 0.3$. The plate is discretized using Bogner–Fox–Schmidt elements.

In Fig. 1b, 1c, and 1d, we illustrate adapted meshes corresponding to three refinement levels. Observe that in this case, wavelets must be added both near the corner and near the point of interest. The convergence of the displacements with increasing levels of refinement is shown in Table 1.

Table 1
Convergence of displacements at point R

Level	DoF	Displacement	Relative % Error
3	260	0.0007348	5.7751
4	292	0.0007747	0.6511
5	324	0.0007757	0.5278
6	348	0.0007760	0.4874
Q_{Ref}	50180	0.0007798	

3.2. Cantilever overhang

In this example, we consider the goal of determining the tip displacements of a uniformly loaded plate with an overhang, as shown in Fig. 2a. The thickness, Young's modulus, and Poisson's ratio are taken, respectively, as $h = 2$, $\mathrm{E} = 1.0 \times 10^7$, and $\nu = 0.3$. From symmetry considerations, we choose a half-symmetric model with the section indicated by line X–X being left unconstrained. Fig. 2b, 2c, and 2d illustrate the adapted meshes at levels $j = 3$, 4, and 5, respectively. The convergence of the tip displacement is shown in Table 2. It can be observed that to estimate accurately the point value at the tip, it suffices to refine only near the origin and not near the point of interest. This is because the detail functions r_j and η_j have large contributions from wavelets only near the origin and not near the point of interest.

Table 2
Convergence of tip displacements

Level	DoF	Displacement	Relative % Error
2	188	2.81636	17.6993
3	212	3.08416	9.8736
4	232	3.25319	4.9338
5	252	3.35341	2.0052
6	272	3.40989	0.3545
Q_{Ref}	33668	3.42203	

4. Summary and conclusions

In this article, we have illustrated how wavelets constructed out of finite-element interpolation functions provide a convenient and unified framework for goal-oriented error estimation and adaptivity. In summary, the most compelling arguments for adopting a consistent multiresolution approach to both error-estimation and mesh refinement are as follows:

1. Adaptive mesh refinement simply amounts to retaining a given subset of basis functions at a finer level of discretization (namely, those that contribute significantly to the functional of interest). In contrast, many contemporary mesh-refinement methods require the imposition of multipoint constraints on irregular nodes that can become extremely cumbersome with multiple levels of refinement.
2. In a unified framework, it is possible to assess the adaptation error at each level using Eq. (3). This is, again, in contrast to classical adaptive finite-element

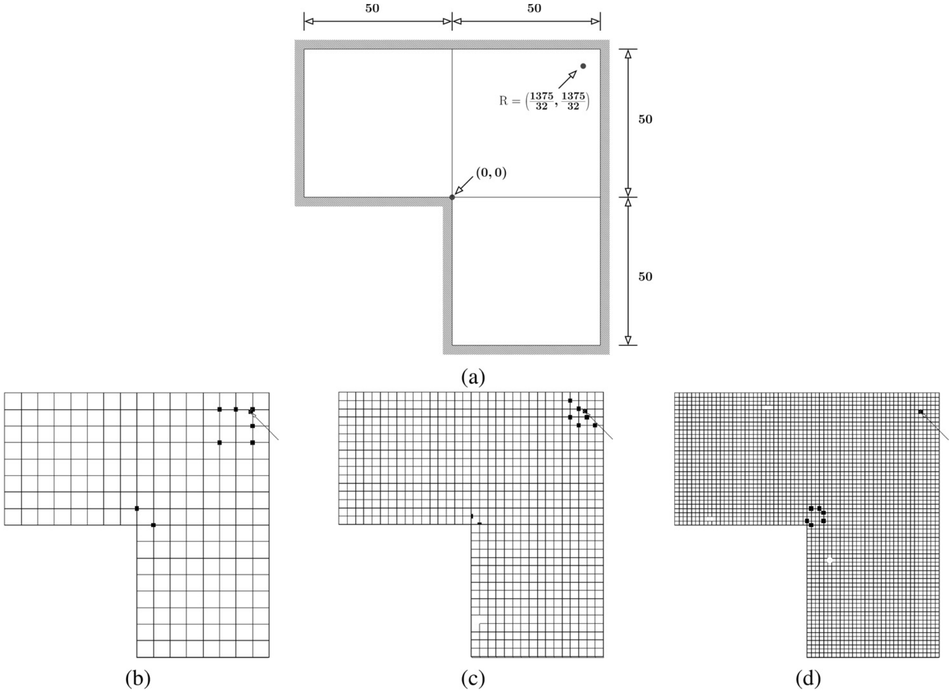

Fig. 1 Point-wise displacements of a clamped L-shaped plate. (a) Problem definition and adapted meshes at level (b) $j = 4$, (c) $j = 5$, and (d) $j = 6$. The arrow indicates the point of interest, R, and the circled vertices denote the set of wavelets retained at each level.

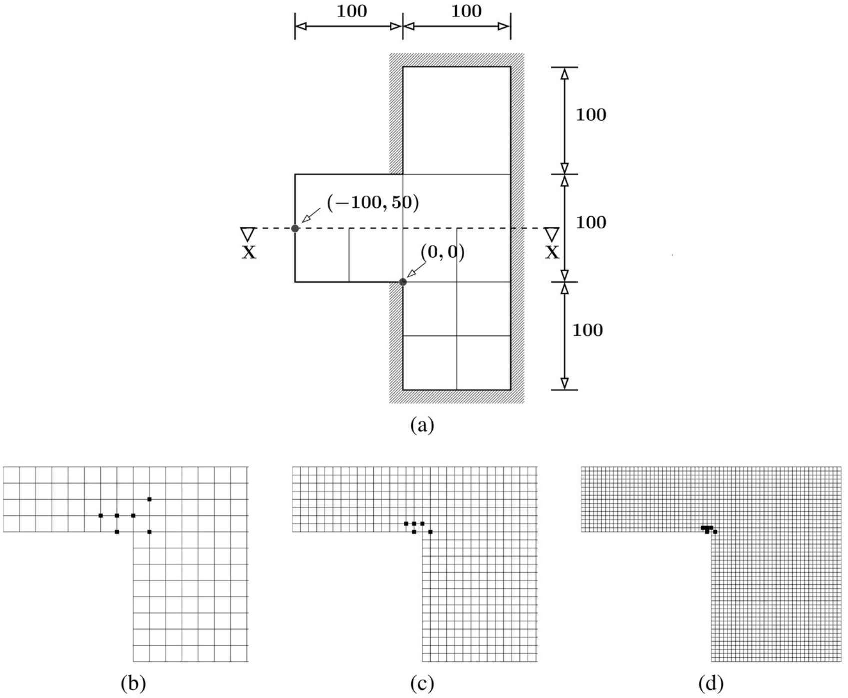

Fig. 2. Tip displacements of a cantilever overhang. (a) Problem definition and adapted meshes at levels (b) $j = 3$, (c) $j = 4$, and (d) $j = 5$.

refinement, where the technique used for error estimation (for instance, post-processing of stress gradients or computing element-level residuals) is often unrelated to the technique used for mesh refinement (remeshing or element subdivision).

References

[1] Yserentant H. On the multilevel splitting of finite element spaces. Numerische Math 1986;49:379–412.
[2] Carnicer J, Dahmen W, Peña J. Local decompositions of refinable spaces. Appl Computat Harmonic Anal 1996;3:125–153.
[3] Becker R, Rannacher R. A feed-back approach to error control in finite element analysis: basic analysis and examples. East–West J Numer Math 1996;4:237–264.
[4] Prudhomme S, Oden JT. On goal-oriented error estimation for elliptic problems: application to the control of pointwise errors. Comput Meth Appl Mech Engng 1999;176:313–331.
[5] Grätsch T, Bathe KJ. A posteriori error estimation techniques in practical finite element analysis. Computers Structures 2005;83(4–5):235–265.
[6] Amaratunga K, Sudarshan R. Multiresolution modeling with operator-customized wavelets derived from finite elements. Comput Meth Appl Mech Engng 2004;submitted.