# A coupled EFGM–BEM approach for dynamic analyses of a halfspace including nonlinear effects

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**Abstract**

A direct approach of coupling the element-free Galerkin method and the boundary element method is presented to investigate the dynamic response of a two-dimensional halfspace. The near field around the transient load is discretized by element-free Galerkin nodes also being able to take into account nonlinear effects, while the far field is modeled by linear boundary elements in order to simulate the radiation conditions at infinity. Ensuring compatibility and equilibrium, the two parts are coupled along their common interfaces, and the resulting system of equations is solved using iterative procedures. It is shown that this methodology is efficient and turns out to be a good alternative to classical methods for complex soil–structure interaction problems.

*Keywords:* Element-free Galerkin method; Boundary element method; Coupling; Nonlinear effects; Iterative procedures; Dynamic analysis; Halfspace

## 1. Introduction

In soil–structure interaction analyses, discretization methods such as the finite element method (FEM) and the boundary element method (BEM) have been proven to be robust and efficient. In this conventional way, the structure is discretized by finite elements, while the soil domain is modeled with either finite or boundary elements. The two parts are coupled together rigorously, ensuring compatibility and equilibrium along the common interface. For the coupling itself, both the direct method [1] and the iterative scheme [2] work well.

The newly appeared mesh-free concepts have shown great promise in challenging traditional mesh-based methods in many fields of computational mechanics. Among all mesh-free methods, the element-free Galerkin method (EFGM) proposed by Belytschko et al. [3] is one of the most popular and best developed. In this paper, an approach of coupling the EFGM and the BEM suggested by Gu et al. [4] is applied to study the dynamic response of a two-dimensional halfspace. The near field around the load is modeled by EFG nodes taking into account nonlinear effects, while the far field is discretized by boundary elements to include Sommerfeld's radiation condition in the infinity. The efficiency of the new methodology, in comparison with the conventional coupled BEM–FEM approach, will be discussed.

## 2. Mathematical formulations and a numerical example

### 2.1. EFGM for nonlinear dynamic analysis

A displacement field around a point **x** generally can be expressed by

$$u^h(\mathbf{x}) = \sum_{j=1}^{m} p_j(\mathbf{x}) a_j(\mathbf{x}) = \mathbf{P}^{\mathbf{T}}(\mathbf{x})\mathbf{a}(\mathbf{x}) \tag{1}$$

where $p_j$ (**x**) is a monomial, $m$ is the number of terms in this monomial, and $a_j$ (**x**) is an unknown coefficient. Applying the numerical treatments summarized by Belytschko et al. [3], $a_j$ (**x**) is obtained by a weighted least-squares fit and the EFG approximation of the displacement field can be expressed by

$$u^{EFG}(\mathbf{x}, t) = u^h = \sum_{I=1}^{n} \phi_I(\mathbf{x}) u_I(t) = \mathbf{\Phi}(\mathbf{x})\mathbf{U_s} \tag{2}$$

where $n$ is the number of nodes inside the compact support of **x**, $\phi_I$ (**x**) is the moving least-squares (MLS)

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shape function, and $u_I$ is the nodal parameter. $\phi_I$ (**x**) is constructed by nodes inside the compact support $\Omega$, and its complexity stems from the ratio of the dimension of $\Omega$ to the minimum nodal distance inside $\Omega$, namely $D_{\max}$. In order to shorten the computation time, $D_{\max}$ is chosen to be 1.2 in contrast to 2.0–2.5 suggested in the literature [3,5]. It will be shown in the following context that such a $D_{\max}$ already leads to a sufficient precision.

Nonlinearities are assumed to appear only in the EFG domain. Applying the procedures stated by Bathe [6], the governing equilibrium equations describing a nonlinear dynamic problem can be given in a finite element (FE) similar form such that

$$\mathbf{M}\ddot{\mathbf{U}}^n_{k+1} + \mathbf{C}\dot{\mathbf{U}}^n_{k+1} + \mathbf{K}_T\Delta\mathbf{U}_{k+1} = \mathbf{F}^n - \mathbf{R}^n_k, \tag{3}$$

$$\mathbf{U}^n_{k+1} = \mathbf{U}^n_k + \Delta\mathbf{U}_{k+1} \tag{4}$$

where **M**, **C**, and $\mathbf{K}_T$ are global mass, damping, and tangential stiffness matrices, respectively. If material damping is neglected, then **C** drops. The matrix $\mathbf{K}_T$ is constructed in the same manner as that of the total Lagrangian FE formulation. However, it is now based on EFG nodes instead of finite elements.

The residual forces $\mathbf{F}^n - \mathbf{R}^n_k$ and the variation of the incremental displacements $\Delta\mathbf{U}_{k+1}$ can be calculated at each iterative step. $\mathbf{U}^n_{k+1}$, $\dot{\mathbf{U}}^n_{k+1}$, and $\ddot{\mathbf{U}}^n_{k+1}$ are the displacement, velocity, and acceleration vectors at time $t_n$ for the iterative step $k+1$, respectively.

The modified Newton–Raphson method is used to solve Eqs (3) and (4) at each time step. To obtain the whole time history, the Newmark integration scheme is applied.

### 2.2 Time domain elastodynamic BEM analysis

For small deformations in homogeneous, isotropic, and linear elastic materials, the equation of motion in two dimensions reads

$$(c_p^2 - c_s^2)u_{k,ki} + c_s^2 u_{i,kk} - \ddot{u}_i + \frac{1}{\rho}b_i = 0, \quad i = 1,2, j = 1,2 \tag{5}$$

where $c_p$ and $c_s$ are dilational and shear wave velocities, respectively. In plane strain state, they are given by

$$c_p = \sqrt{\frac{E(1-\nu)}{\rho(1+\nu)(1-2\nu)}} \quad \text{and} \quad c_s = \sqrt{\frac{E}{2\rho(1+\nu)}} \tag{6}$$

where $E$ is the elastic modulus, $\nu$ is Poisson's ratio, and $\rho$ is the material density.

Assuming zero body forces $b_j$ and zero initial conditions, Eq. (5) can be reduced to an integral equation of the form

$$c_i u_i(\xi, t) = \int_\Gamma \int_0^t [u_j^* s_j(x,\tau) - s_j^* u_j(x,\tau)]d\tau d\Gamma \tag{7}$$

where $s_j$ $(x,\tau)$ and $u_j$ $(x,\tau)$ represent tractions and displacements, respectively, at the surface $\Gamma$ at time $\tau$. $u_i$ $(\xi,t)$ is the displacement at an arbitrary point $\xi$ located either on the smooth surface ($c_i = 0.5$) or in the interior of the body ($c_i = 1.0$). $s_j^*$ is the fundamental solution for the traction, which corresponds to $u_j^*$, the fundamental solution for the displacement. More details are provided by Beskos [7].

Eq. (7) is discretized and solved numerically. Constant elements are used for the spatial discretization, while linear and constant shape functions are applied to describe the time dependence of the displacements and the tractions, respectively. Finally, collocation at each boundary node and at all time steps leads to a system of equations

$$\mathbf{U}^l\mathbf{s}^N = \mathbf{T}^l\mathbf{u}^N + \sum_{m=2}^{N} \mathbf{T}^m\mathbf{u}^{N-m+l} - \mathbf{U}^m\mathbf{s}^{N-m+1} \tag{8}$$

in which $\mathbf{U}^l$ and $\mathbf{T}^m$ are the influence matrices of the system at time $m\Delta t$. To obtain consistency between the EFG and boundary element (BE) formulations, the boundary tractions **s** have to be transformed to a force vector **F** by means of a matrix **M**, such that

$$\mathbf{F} = \mathbf{M}\mathbf{s} \tag{9}$$

**M** can be evaluated by the BE shape functions; more details are given by von Estorff and Prabucki [8].

### 2.3 Coupling procedure

The coupling of the BEM and the EFGM is accomplished by introducing interface elements between both subdomains (Fig. 1). In such interface elements, a hybrid displacement function is defined. Along the interface boundary $\Gamma_I$, compatibility and equilibrium conditions must be satisfied, i.e.

$$\mathbf{u}_I^{(1)} = \mathbf{u}_I^{(2)}, \ \mathbf{F}_I^{(1)} + \mathbf{F}_I^{(2)} = 0 \tag{10}$$

The modified displacement approximation in $\Omega_1$ has the form

$$u_i^h(\mathbf{x},t) = \begin{cases} [1 - R(\mathbf{x})]u^{BE}(\mathbf{x},t) + R(\mathbf{x})u^{EFG}(\mathbf{x},t)\,, & \mathbf{x} \in \Omega_I \\ R(\mathbf{x})u^{EFG}(\mathbf{x},t)\,, & \mathbf{x} \in \Omega_1 - \Omega_I \end{cases} \tag{11}$$

where $u_i^h$ is the displacement of a point in $\Omega_1$, $u^{EFG}$ (**x**,$t$) is the EFG displacement given by Eq. (2), and $u^{BE}$ (**x**,$t$) is the BE displacement given by

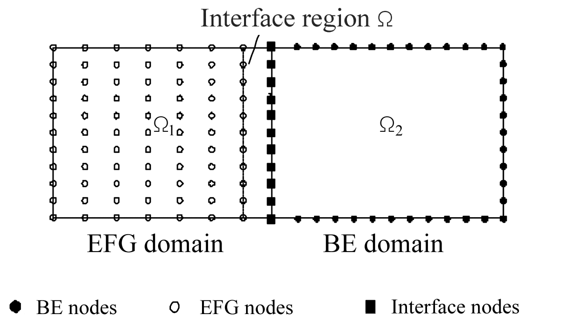


Fig. 1. Coupling of an EFGM ($\Omega_1$) and a BEM ($\Omega_2$) subdomain.

$$u^{BE} = \sum_{I=1}^{n_e} N_I(\mathbf{x}) u_I(t) \tag{12}$$

where $N_I$ ($\mathbf{x}$) is the BE shape function and $n_e$ is the number of nodes in an element.

The ramp function $R(\mathbf{x})$ is equal to the sum of the BE shape functions associated with interface element nodes that locate on the interface boundary, i.e.

$$R(\mathbf{x}) = \sum_{I}^{j} N_I(\mathbf{x}), \quad \mathbf{x}_I \in \Gamma_I \tag{13}$$

where $j$ is the number of nodes located on $\Gamma_I$. According to the property of the BE shape function, $R$ is equal to one along $\Gamma_I$ and vanishes in domains other than $\Omega_I$

$$R(\mathbf{x}) = \begin{cases} 1 & x \in \Gamma_I \\ 0 & x \in \Omega_1 - \Omega_I \end{cases} \tag{14}$$

The interface shape function can then be developed by substituting the displacement approximations from Eqs (2) and (12) into Eq. (11); one obtains

$$u_i^h(\mathbf{x}, t) = \sum_{I=1}^{n} \tilde{N}_I(\mathbf{x}) u_{iI}(t) \tag{15}$$

where the interface shape function $\tilde{N}_I(\mathbf{x})$ reads

$$\tilde{N}_I(\mathbf{x}) = \begin{cases} [1 - R(\mathbf{x})] N_I(\mathbf{x}) + R(\mathbf{x})\phi_I(\mathbf{x}) & \mathbf{x}_I \in \Omega_I \\ R(\mathbf{x})\phi_I(\mathbf{x}) & \mathbf{x}_I \in \Omega_1 - \Omega_I \end{cases} \tag{16}$$

The derivative of the interface shape function is

$$\tilde{N}_{I,i}(\mathbf{x}) = \begin{cases} [1 - R(\mathbf{x})] N_{I,i}(\mathbf{x}) + R_{,i}(\mathbf{x})\phi_I(\mathbf{x}) + R(\mathbf{x})\phi_{I,i}(\mathbf{x}) \\ \quad - R_{,i}(\mathbf{x}) N_I(\mathbf{x}) \quad \mathbf{x}_I \in \Omega_I \\ R_{,i}(\mathbf{x})\phi_I(\mathbf{x}) + R(\mathbf{x})\phi_{I,i}(\mathbf{x}) \quad \mathbf{x}_I \in \Omega_1 - \Omega_I \end{cases} \tag{17}$$

The approximation based on the above interface shape function is compatible and reproduces the linear field exactly [9]. The system equations of the BE subdomain and the EFG subdomain can thus be assembled together.

### 2.4. Numerical example

The current approach is applied to investigate the dynamic response of a halfspace. Its geometry and discretization are depicted in Fig. 2. In the EFG subdomain, a perfectly plastic material obeying the Mohr–Coulomb yielding criterion is assumed. The necessary material properties are density $\rho = 3150\,\text{kg/m}^3$, elastic modulus $E = 1.77 \times 10^3$ MPa, Poisson's ratio $\nu = 0.25$, cohesion $C = 12.5$ MPa, and the internal friction angle $\theta = 10°$. The dilational and shear wave velocities are $c_1 = 821$ m/s and $c_2 = 474$ m/s,

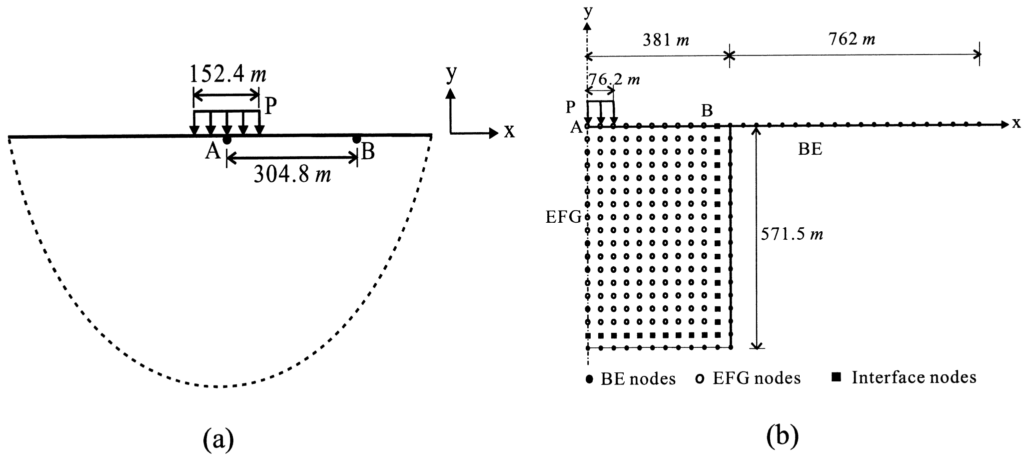


Fig. 2. (a) The geometry and loading situation of the halfspace. (b) The EFG and BE discretizations of the right-hand half of the problem domain.

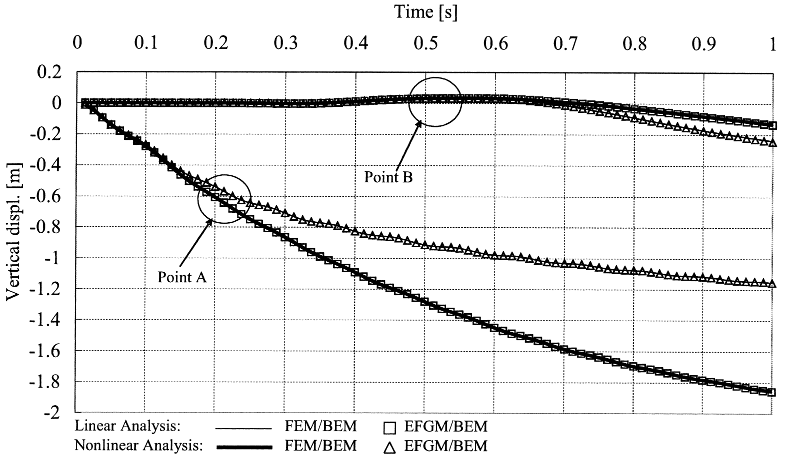


Fig. 3. Computational results from the current coupled EFGM–BEM approach in comparison with those from the conventional coupled FEM–BEM approach.

respectively. A transient uniform compressive stress that is symmetrical about point A is applied on the free surface of the halfspace. The load is given by $q(t) = 68.96H(t-0)$ MPa, where $H$ is the Heaviside step function. Since the model is symmetrical about point A, Fig. 2b shows only the right-hand half of the model.

The vertical displacements at points A and B of the surface are computed using coupled EFG–BE and FE–BE models, respectively. In the second model, the FE discretization with bilinear quadrilateral elements is set to be the same as that of the EFG background mesh in the first model. The size of the time step for both models is set to 0.0125 s. Linear as well as nonlinear analyses are carried out for both models. Note again that in nonlinear analysis, elastoplastic effects are restricted only in the EFG or FE domain. Computational results are depicted in Fig. 3. The nonlinear effect in point A is more significant compared with that in point B. Excellent agreement of the results obtained by both numerical methods is recognized. The current $D_{\max} = 1.2$ in the EFG domain seems absolutely sufficient for the computation. Using a Pentium 4 running at 2.8GHz with 1 GB RAM, the CPU time of the coupled EFG–BE method with $D_{\max} = 1.2$ is around 10% less than that of the coupled FE–BE method in the nonlinear analysis.

## 3. Conclusions

A coupled EFGM–BEM approach is proposed to study the dynamic response of a halfspace under transient loads. Nonlinear effects assumed to occur in the EFG domain are included in the analysis. Transient boundary elements are used to ensure that the radiation of waves to the infinity is taken into account. A relatively small $D_{\max}$ in the EFG domain leads to sufficient accuracy. The computational effort with such a $D_{\max}$ in the current approach is less than that of the conventional coupled BE–FE method. The coupled EFGM–BEM approach has the potential to investigate complex soil–structure interactions with local nonlinearities.

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