Mach-uniformity through the coupled pressure- and temperature-correction algorithm

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Abstract

We present a new type of algorithm: the coupled pressure- and temperature-correction algorithm. This finds its place in between the fully coupled and the fully segregated approaches, and it is constructed such that Mach-uniform accuracy and efficiency are obtained. The essential idea is the separation of the convective and the acoustic/thermodynamic phenomena: a convective predictor is followed by an acoustic/thermodynamic corrector. For a general case, the corrector consists of a coupled solution of the energy and the continuity equations for both pressure and temperature corrections. For the special case of an adiabatic perfect gas flow, the algorithm reduces to a fully segregated method, with a pressure-correction equation based on the energy equation. Various test cases are considered, which confirm that Mach-uniformity is obtained.

Keywords: Mach-uniform; Segregated; Coupled; Pressure-correction

1. Introduction

Mach-uniform algorithms are indispensible tools in numerous flow situations. With preconditioning, the originally high-speed density-based algorithms were extended towards the low Mach number regime [1,2]. Several attempts have been made to develop compressible pressure-correction methods [3–11] but they are not satisfying: either they are not Mach-uniform or they are not applicable in general flow situations. In this paper, we aim to construct a collocated pressure-correction method that does have these features. Analysis of the Euler and Navier–Stokes equations, valid for a general fluid, reveals how to reach Mach-uniformity. For a general case, this leads to an algorithm that finds its place in between the fully coupled and the fully segregated approaches: the coupled pressure- and temperature-correction algorithm.

2. Mach-uniform accuracy

The governing Navier–Stokes equations are non-dimensionalized by choosing reference quantities for pressure, temperature, and length scale. A collocated vertex-centered finite volume method and a first-order time integration scheme are applied to discretize them. The conductive heat flux and the viscous terms are discretized centrally. The spatial flux definitions are advection upwind splitting method (AUSM)+ [12], where the transported quantities are first-order upwinded. At low Mach numbers, special measures have to be taken with respect to the scaling and decoupling problem [13]. Doing so enables us to reach Mach-uniform accuracy.

3. Mach-uniform efficiency

Mach-uniform efficiency implies a good convergence rate, whatever the Mach number is. The low speed limit is especially critical, due to severe time-step restrictions imposed by the stability of the scheme. They can be of an acoustic as well as of a diffusive nature.

The first restriction is expressed by the acoustic Courant–Friedrichs–Lewy (CFL) number $CFL_{u+c} = (u + c)\Delta t/\Delta x$. To remedy the low Mach stiffness problem, the acoustic CFL limit has to be removed. This can be done by treating the terms that carry acoustic information in an implicit way. Considering the conservative Euler equations, the question raises how the acoustic

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terms can be identified. This becomes clear by transforming the equations into a quasi-linear set:

$$p_t + up_x + \underline{\rho c^2 u_x} = 0 \tag{1}$$

$$u_t + uu_x + \underline{p_x/\rho} = 0 \tag{2}$$

$$s_t + us_x = 0 \tag{3}$$

The underlined terms represent the acoustic part of the system. Using the equations of state $\rho = \rho(p, T)$ and $\rho e = \rho e(p, T)$, and a general definition for the speed of sound c, the pressure Eq. (1) can be written as

$$[\rho_p(\rho e)_T - (\rho e)_p \rho_T]\,(p_t + up_x) + (\rho e)_T \underline{\rho u_x} - \rho_T \underline{(\rho e + p) u_x} = 0 \tag{4}$$

Note that the first underlined term finds its origin in the continuity equation, while the second comes from the energy equation. Returning to the conservative Euler equations gives

$$\rho_t + \underline{(\rho u)_x} = 0 \tag{5}$$

$$(\rho u)_t + (\rho uu)_x = -\underline{p_x} \tag{6}$$

$$(\rho E)_t + (\rho Hu)_x = 0 \tag{7}$$

with $\rho Hu = (\rho e + p)u + \rho u^2 u/2$. The underlined terms will be treated implicitly in order to remove the acoustic CFL limit. In the static enthalpy flux, only the velocity u has to be treated implicitly, since only this variable appears under a derivative in the acoustic term.

At this point, it is very instructive to consider some special cases. If the density ρ can be considered as constant, then the second acoustic contribution in Eq. (4) disappears. It means that for this case, the energy equation contains no acoustic information. Furthermore, the pressure equation, Eq. (4), originates from the continuity equation alone and reduces to the incompressible constraint $u_x = 0$. On the other hand, for a perfect gas, ρe depends only on the pressure. The first acoustic part in Eq. (4) therefore becomes zero, so that the continuity equation doesn't contain acoustic information. The pressure Equation, Eq. (4), is derived from the energy equation alone. We stress, however, that this does not hold when conductive temperature terms occur in the energy equation.

The diffusive time step limit is expressed by the Von Neumann number $Ne = \alpha \Delta t / \Delta x^2$, with α being the thermal diffusivity or the kinematic viscosity. To remove this, the diffusive conductive and viscous terms have to be treated implicitly.

4. Implementation

The first step is a predictor step for density and momentum, from the continuity and momentum equations, respectively. The predicted values ρ^*, $(\rho u)^*$, together with the old pressure values, determine an intermediate state *. The predictor step takes place under a frozen pressure and in essence determines a velocity u^*. It therefore can be considered as a convective step.

Secondly, corrections with respect to the predictor values are defined. They are introduced into the discretized equations, namely in the terms at the new time level. Amongst them are the implicit diffusive and acoustic terms identified in Section 3. The corrector step therefore can be considered as an acoustic/thermodynamic step.

In the continuity equation, the density is expanded as $\rho^{n+1} = \rho^* + \rho^*_p p' + \rho^*_T T'$. The mass flux, which is an acoustic term, is corrected as $(\rho u)^{n+1} = (\rho u)^* + (\rho u)'$, and the momentum correction is related to pressure corrections through the momentum equation. The continuity equation thus becomes an equation for both pressure and temperature corrections.

In the energy equation, the total energy is expanded as $(\rho E)^{n+1} = (\rho E)^* + (\rho e)^*_p p' + (\rho e)^*_T T'$. According to the identification of the acoustic terms, the enthalpy flux is corrected as $\rho Hu = (\rho e + p)^* [(\rho u)^* + (\rho u)']/\rho^* + (\rho u^2)^* u^*/2$, and the momentum correction is replaced. In the heat flux, temperature corrections are introduced to avoid a diffusive time step limit, $q^{n+1} = \kappa(T^* + T')_x$.

Introducing these terms into the energy equation results in a second equation containing both pressure and temperature corrections. The two correction equations, originating from the continuity and energy equations, are solved in a coupled way. We therefore refer to this method as the coupled pressure- and temperature-correction algorithm. After updating the pressure and the temperature, momentum is updated through the momentum equation. Density is updated through the equation of state.

For a perfect gas, in the absence of heat transfer, all temperature corrections disappear from the energy equation. Thus, no coupled solution with the continuity equation is needed: after the predictor step, pressure corrections can be determined from a pressure-correction equation based on the energy equation. However, when heat conduction is present, a coupled solution is needed due to the temperature corrections in the heat flux.

5. Results

5.1. Adiabatic flow

First, the Euler equations are considered, i.e. heat conduction and viscosity are neglected. As a test case, we took a one-dimensional nozzle flow of a perfect gas.

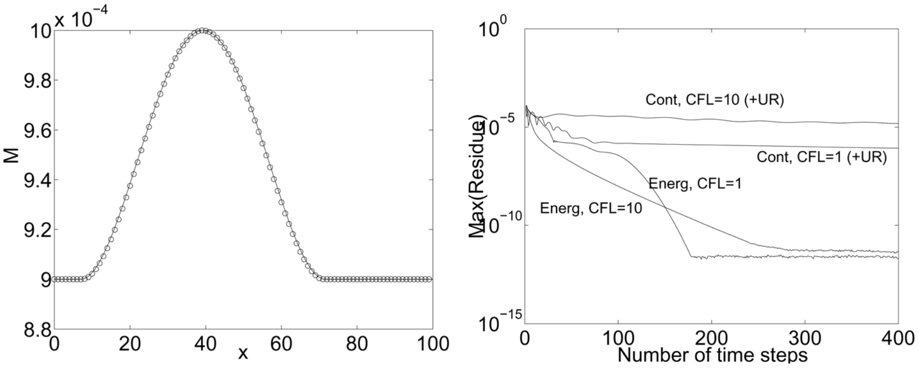

Fig. 1. Subsonic nozzle flow, $M_t = 0.001$. (a) Mach number distribution. (b) Convergence plot, $CFL_u = 1$ and 10. *Energ*: Mach-uniform algorithm with pressure-correction equation based on the energy equation. *Cont*: equivalent algorithm based on the continuity equation (computation with underrelaxation, UR).

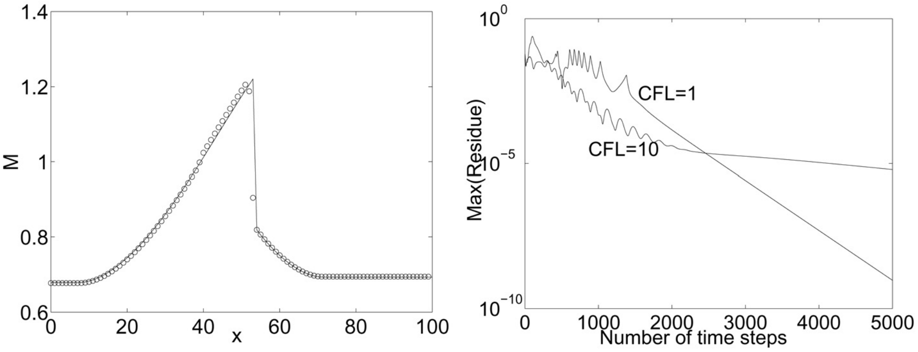

Fig. 2. Transonic nozzle. (a) Mach number distribution. (b) Convergence plot, $CFL_u = 1$ and 10.

The fully segregated algorithm with a pressure-correction equation based on the energy equation was used.

As a low-speed test, we consider a subsonic nozzle flow with a throat Mach number M_t of 10^{-3}. The time step is calculated from a chosen convective CFL number, $\Delta t/\Delta x = CFL_u/max(u)$. This corresponds with an acoustic CFL_{u+c} number that is about 1000 times higher. The CFL_u number could be taken to be arbitrarily high: there is no stability limit. Fig. 1 shows the results for both the Mach-uniform algorithm and an equivalent algorithm where the pressure-correction equation was derived from the continuity equation.

The continuity-based algorithm has a very bad convergence rate at this low Mach number. Furthermore, the computations could be made stable only under a severe underrelaxation. It clearly suffers from the stiffness problem, which results in a convergence breakdown. The Mach-uniform algorithm, on the other hand, performs very well for this low-speed flow, with regard to accuracy as well as efficiency.

Also for the high-speed case of a transonic nozzle, there is no CFL limit. Fig. 2 shows the Mach number distribution and the convergence plot. Again, a good convergence rate and accuracy are obtained. Clearly, the algorithm shows Mach-uniform efficiency and accuracy.

5.2. Nonadiabatic flow

First, we consider again the test cases of a one-dimensional nozzle flow, but now heat conduction is taken into account. Just as for the adiabatic computations, no acoustic time-step limit occurs. All the simulations are done at a convective CFL_u number of 1. If a diffusive time-step limit is to be avoided, the coupled solution of energy and continuity equation for both pressure and temperature corrections is needed. Table 1 shows the stability results for different nozzle flows. The columns indicated with *exp* refer to an explicit calculation of the heat flux, which is added as an extra term in the right-hand side of the pressure-correction equation based on the energy equation. *Coup* refers to the coupled pressure and temperature correction method. Different

Table 1
Exp: pressure-correction equation based on the energy equation, heat flux calculated with T^*. *Coup*: coupled pressure- and temperature-correction method

		Stable?				Stable?	
κ	*Ne*	*exp*	*coup*	κ	*Ne*	*exp*	*coup*
10^{-5}	2.42×10^{-4}	Yes	Yes	10^{-5}	2.42×10^{-6}	Yes	Yes
0.01	0.242	Yes	Yes	1	0.242	Yes	Yes
0.1	2.42	No	Yes	10	2.42	No	Yes
1	24.2	No	Yes	100	24.2	No	Yes
	Subsonic $M_t = 0.01$				**Transonic**		

values for the nondimensional conduction coefficient κ were tested. The results show clearly that the method with explicit heat flux becomes unstable as soon as the Von Neumann number becomes higher than order unity. The coupled method, however, stays stable, no matter how high κ is taken.

As a second test case for nonadiabatic flow, the two-dimensional thermal driven cavity problem is considered [14], with $Ra = 10^3$ and $\in = 0.6$. This is a very challenging problem for our algorithm. Indeed, due to the very low Mach numbers, no acoustic CFL limit is allowed. Furthermore, in the vicinity of the wall, the convective speeds are very small and the conduction becomes the dominating phenomenon. Therefore, a diffusive Von Neumann limit is to be avoided, so that a coupled solution of the continuity and the energy equation is needed. Streamline patterns, temperature contours, and a convergence plot are shown in Fig. 3. From the values used for α, Δt, and grid dimensions, we obtain for the maximum CFL_{u+c} and Neumann number $O(10^7)$ and $O(10^4)$, respectively. Clearly, the acoustic and diffusive time-step limits have been removed.

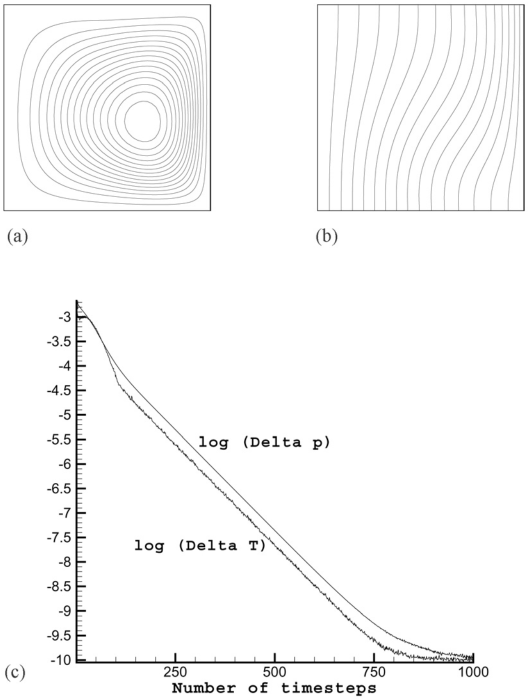

Fig. 3. Thermal driven cavity. (a) Streamline patterns. (b) Temperature contours. (c) Convergence plot.

6. Conclusion

We have presented a new type of algorithm: the coupled pressure- and temperature- correction algorithm. The essential idea is the separation of the convective phenomenon on the one side and of the acoustic/thermodynamic phenomenon on the other side. Based on a theoretical analysis, the algorithm was constructed so that Mach-uniform accuracy and efficiency are obtained, which was confirmed by the test results.

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