The Fourier boundary element method and its singularities

Fabian Duddeck

BMW Group, EK-212, Center of Research and Innovation, Knorrstrasse 147, Munich D-80636, Germany

Abstract

One of the limitations of boundary element methods (BEM) lies in their need for a fundamental solution. In many engineering problems, this function is not known analytically but constructed numerically. The corresponding pre-computed values are stored in tables and later – during the computation – the required values are interpolated. To overcome this drawback and to accelerate the computation of the BEM, a Fourier transformed boundary element method was proposed. The focus of this paper is the treatment of singular and hypersingular integrals of this Fourier BEM. It can be shown easily that all strong and hypersingular values cancel. The computation of the singular integrals is hence straightforward in the Fourier space and can be used in traditional BEM approaches.

Keywords: Boundary element method; Engineering problems; Singular and hypersingular integrals; Distribution theory

1. Introduction

For boundary element methods (BEM), it is crucial to know the fundamental solution (or Green's function), which is the response of the infinite medium to a unit force [1]. Particular boundary conditions are generally not specified but may be accounted for if advantageous; this distinguishes fundamental solutions from Green's functions. Both may be interpreted as the inverse of the differential operator. For general nonlinear cases, a fundamental solution cannot be found. Thus, BEM treating nonlinear phenomena are referring to the corresponding linear cases; the nonlinear terms are shifted to the right-hand side of the equation [2]. Thus, nonlinear BEM also require linear fundamental solutions.

Nevertheless, in linear problems, it is not evident to find a fundamental solution. For differential operators with nonconstant coefficients, this is possible only in particular cases; a general approach is not established yet, but first ideas may be taken from Pomp [3]. Even in cases with constant coefficients, the fundamental solution often is not known due to the complexity of the differential operator. For example, there is no fundamental solution for arbitrary anisotropic three-dimensional elastic media.

After a Fourier transform of the differential equation with respect to all coordinates, i.e. with respect to time and space, the inversion of the differential operator is always possible without difficulty, as long as the differential operator is linear and has constant coefficients. For elasticity problems, this leads to an inversion of a simple 3×3 matrix. The drawback lies then in the inverse Fourier transform, which is normally for static problems a three-dimensional integral transform and for dynamic cases a four-dimensional integration. These integrations can only be computed analytically in special cases. Thus, the fundamental solution is normally known only in the transformed space. In the original space, one has to compute the inverse transform numerically, storing the results in tables before the BEM analysis. The values required later in the BEM are interpolated between the initially established numerical values. Thus, there is the error introduced by the truncation of the numerical inverse Fourier transform and the error originating from interpolation. Unfortunately, fundamental solutions are singular near the origin; for BEM, singular and hypersingular integrals have to be computed. For these, the errors have a remarkable influence and one has to be careful in the implementation of this approach based on pre-evaluated numerical fundamental solutions.

To overcome this drawback, the author has proposed an alternative approach [2]. No numerical inverse Fourier transform is required; the method is based only on the Fourier transformed fundamental solution, which is available in all linear cases with constant coefficients. Thus, the approach generalizes the BEM to a large field of engineering applications. All computations are done in the transformed space. For this, the test and trial

Tel.: +49 89 382 29213; Fax: +49 89 382 42820; E-mail: fabian.duddeck@bmw.de

Computational Fluid and Solid Mechanics 2005
K.J. Bathe (Editor)

functions have to be transferred to the Fourier space, which is easily possible for straight elements. A Galerkin method is preferred to the collocation version of BEM because the former leads to symmetric matrices and hence reduces the numerical effort. The normally required double integration is replaced by a simple integration. The main theoretical background is Parseval's theorem stating the equivalency of scalar products in the original space and the Fourier transformed space. The integrals of the Galerkin BEM are either convolution integrals or scalar products. The Fourier transform converts convolution to multiplication – this integration is vanishing – and scalar products to scalar products. The matrices of the Fourier BEM are identical to those obtained by the traditional BEM approach.

One of the important items of BEM is the computation of singular values [4]. For collocation methods, weak singular and strong singular integrals have to be solved. In some particular cases, e.g. fracture mechanics, even hypersingular integrals occur. Weak singularities can be integrated directly; for strong and hypersingular integrals, a regularization procedure has to be established to handle Cauchy and Hadamard principle values (finite parts) [5,6]. A more rigorous study of the integral equations used in BEM shows that all non-integrable singularities occur on the left-hand side as well as on the right-hand side of the equations. Thus, the singular terms cancel one another. This is shown easily by the distributional derivation of the integral equations presented in Chapter 3 of Duddeck [2].

In this contribution, this distributional approach is presented and developed particularly for the treatment of singular integrals. The singular layer potential, double layer potential, and hypersingular potential are derived and their equivalent expressions in the Fourier transformed space are given. As examples, the Poisson equation, the Kirchhoff equation, and the Lamé equation are considered. The singularities in the Fourier transformed equations required for the Fourier BEM differ from those in the original space. Thus, a new strategy to solve them is presented, which develops the ideas to be found in Duddeck [2]. Especially the hypersingular integrals can be computed directly in the Fourier space. The obtained values are identical to those that would be obtained in the original space. Thus, the method presented here may also be of interest for solving singular integrals in standard BEM.

2. The principle of Fourier BEM

The principle of Fourier BEM is presented here for the Poisson equation. It can be transferred easily to all linear differential equations if the coefficients are constant. Scalar and vectorial problems can be treated. The differential problem is

$$\begin{aligned} \Delta u(x) &= -f(x); && x \in \Omega, \\ u(x) &= u_\Gamma(x); && x \in \Gamma_u \subset \partial\Omega, \\ t(x) &= t_\Gamma(x); && x \in \Gamma_t \subset \partial\Omega, \end{aligned} \tag{1}$$

where $u(x)$, $f(x)$, $u_\Gamma(x)$, and $t_\Gamma(x)$ are the unknown quantity (e.g. temperature), the volume sources, and the Dirichlet and Neumann boundary conditions, respectively. Δ is the Laplacian. The system (single and double layer potentials) of the Galerkin boundary integral equations is [4]:

$$\begin{aligned} \int_{\Gamma_x} \phi_t^j(x)\kappa(x)u(x)\mathrm{d}\Gamma_x &= \int_{\Gamma_x} \phi_t^j \int_{\Omega} f(y)U(x-y)\mathrm{d}\Omega_y\mathrm{d}\Gamma_x \\ &+ \sum_i^{N_i} \mathrm{t}_i \int_{\Gamma_x} \phi_t^j(x) \int_{\Gamma_y} \phi_t^i(y)U(x-y)\mathrm{d}\Gamma_y\mathrm{d}\Gamma_x \\ &- \sum_i^{N_i} \mathrm{u}_i \int_{\Gamma_x} \phi_t^j(x) \int_{\Gamma_y} \phi_u^j(y)A_t^i U(x-y)\mathrm{d}\Gamma_y\mathrm{d}\Gamma_x \end{aligned} \tag{2}$$

$$\begin{aligned} \int_{\Gamma_x} \phi_u^j(x)A_t^j\{\kappa(x)u(x)\}\mathrm{d}\Gamma_x &= \int_{\Gamma_x} \phi_u^j(x) \int_{\Omega} f(y)A_t^i U(x-y)\mathrm{d}\Omega_y\mathrm{d}\Gamma_x \\ &- \sum_i^{N_i} \mathrm{t}_i \int_{\Gamma_x} \phi_u^j(x) \int_{\Gamma_y} \phi_t^i(y)A_t^i U(x-y)\mathrm{d}\Gamma_y\mathrm{d}\Gamma_x \\ &- \sum_i^{N_i} \mathrm{u}_i \int_{\Gamma_x} \phi_u^j(x) \int_{\Gamma_y} \phi_u^i(y)A_t^i A_t^i U(x-y)\mathrm{d}\Gamma_y\mathrm{d}\Gamma_x \end{aligned} \tag{3}$$

where $U(x)$ is the fundamental solution, A_t is the differential operator $-\partial = -\nu\nabla$ at the boundary, $k(x)$ is the free term (normally ½ for smooth boundaries), and ϕ_u and ϕ_t are the test and trial functions for the temperature and the flux at the boundaries. For the Fourier transform of these equations, all integrals are extended artificially to infinity by assuming zero outside the support of the test functions [2]. Two theorems are required:

$$\text{Theorem of Parseval: } \langle a(x), b(x)\rangle = \frac{1}{(2\pi)^n}\left\langle \hat{a}(-\hat{x}), \hat{b}(\hat{x})\right\rangle \tag{4}$$

$$\text{Convolution theorem: } a(x) * b(x) \overset{Fourier}{\longleftrightarrow} \hat{a}(\hat{x})\hat{b}(\hat{x})$$

$(\hat{\ })$ denotes a Fourier transformed quantity. The integrals in Eqs (2) and (3) can be interpreted as convolutions and scalar products. Thus, they can be converted to:

$$\begin{aligned} \left\langle \hat{\phi}_t^j(-\hat{x}), \hat{u}_\chi(\hat{x})\right\rangle &= \left\langle \hat{\phi}_t^j(-\hat{x}), \hat{f}_\chi(\hat{x})\hat{U}(\hat{x})\right\rangle \\ &+ \sum_i^{N_i} \mathbf{t}_i \left\langle \hat{\phi}_t^j(-\hat{x}), \hat{\phi}_t^i(\hat{x})\hat{U}(\hat{x})\right\rangle - \sum_i^{N_i} \mathbf{u}_i \left\langle \hat{\phi}_t^j(-\hat{x}), \hat{\phi}_u^i \hat{\mathbf{A}}_t^i \hat{U}(\hat{x})\right\rangle, \end{aligned}$$

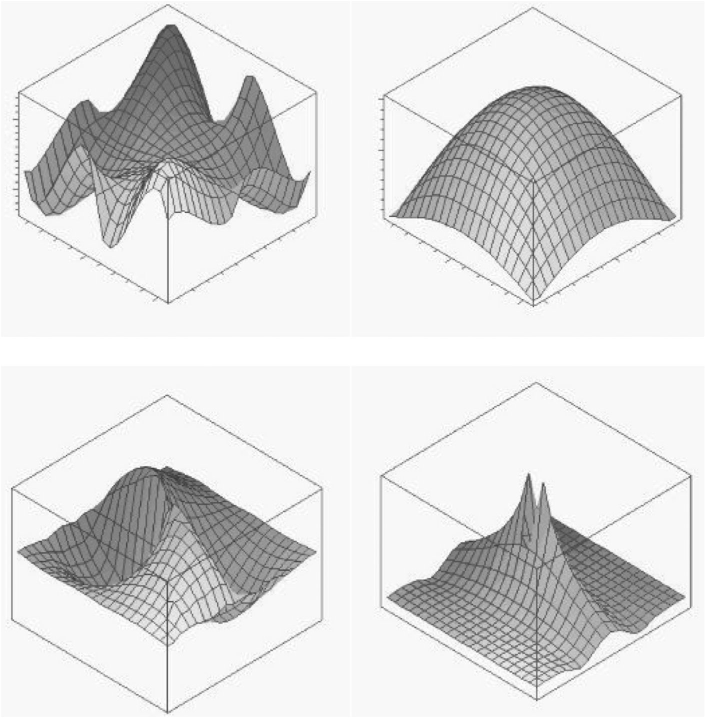

Fig. 1. Perpendicular and parallel displacements of an isotropic (upper row) and anisotropic (lower row) linear elastic medium to a uniform loading (upper row) and a single unit force in the middle of the square (lower row) computed by Fourier BEM [2].

$$
\begin{aligned}
-\left\langle \hat{\phi}_u^j(-\hat{x}), \hat{\mathbf{A}}_t^i \hat{u}_\chi(\hat{x}) \right\rangle = &-\left\langle \hat{\phi}_u^j(-\hat{x}), \hat{f}_\chi(\hat{x}) \hat{\mathbf{A}}_t^i \hat{U}(\hat{x}) \right\rangle \\
&- \sum_i^{N_i} \mathbf{t}_i \left\langle \hat{\phi}_u^j(-\hat{x}), \hat{\phi}_t^i(\hat{x}) \hat{\mathbf{A}}_t^i \hat{U}(\hat{x}) \right\rangle \\
&+ \sum_i^{N_i} \mathbf{u}_i \left\langle \hat{\phi}_u^j(-\hat{x}), \hat{\phi}_u^i \hat{\mathbf{A}}_t^i \hat{\mathbf{A}}_t^i \hat{U}(\hat{x}) \right\rangle
\end{aligned} \tag{5}
$$

These Fourier transformed boundary integral equations are totally equivalent to those of the traditional approach. The unknown boundary coefficients u_i and t_i are identical to those of standard BEM; they are obtained from the matrix equations either in the original or in the Fourier space. No inverse transform is required. The main advantage is that now only the Fourier transformed fundamental solution is used, which can be obtained easily.

3. Two examples: two-dimensional elastic continuum and Kirchhoff plate

The approach was applied to a two-dimensional problem from linear elasticity (isotropic and anisotropic) and to a simple plate problem (Kirchhoff theory). The results are shown in Figs 1 and 2. In all cases, a square was chosen as the geometry. A single unit force or uniform loading was applied. In Fig. 1, the effect of the anisotropy is clearly visible. The unit force in the middle of the square in the case of anisotropic elasticity is leading to an infinite response in the direction of the force at the location of the force. The examples were computed analytically, and thus the infinite character is represented correctly.

4. Weak, strong, and hypersingular integrals for BEM and Fourier–BEM

A local singularity, e.g. a Dirac distribution, is converted by the Fourier transform to a global singularity, i.e. a nonvanishing behavior at infinity. The Dirac is transformed to a constant. Thus, the singularities occurring in traditional BEM still exist after reformulating the problem by the Fourier BEM concept. The shift from local to global and vice versa requires a new treatment of these singularities. For the sake of brevity, the presentation here is limited to the worst case: the computation of the hypersingular entries. The other singularities can be handled analogously [2].

A hypersingular matrix entry is, for example, obtained from

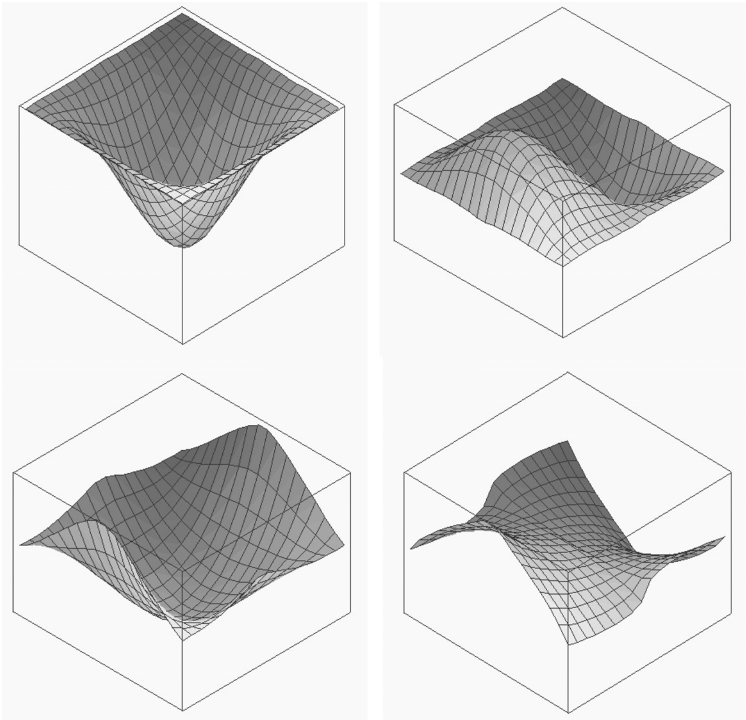

Fig. 2. Isotropic elastic Kirchhoff plate (thin clamped plate) under uniform loading. Upper row: vertical displacements on the left and slope φ_1 on the right. Lower row: moment m_{11} (left) and shear force q_1 (right) computed by Fourier BEM [2].

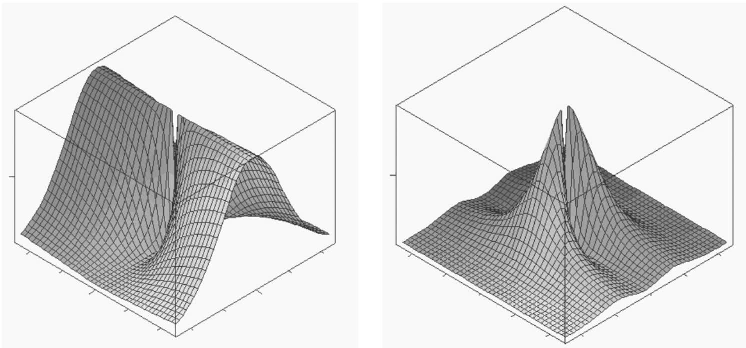

Fig. 3. Integrand of the hyper singular entry in the Fourier space (left) and the same term after accounting for all singular terms of the boundary integral equation [2].

$$G^{ij} = \left\langle \phi_u^j, \phi_u^i * \mathbf{A}_t^i \mathbf{A}_t^i U \right\rangle = \left\langle \hat{\phi}_u^j(-\hat{x}), \hat{\phi}_u^i(\hat{x}) \hat{\mathbf{A}}_t^j \hat{\mathbf{A}}_t^i \hat{U}(\hat{x}) \right\rangle \quad (6)$$

The first term is the integral in the original space; the second term is the expression in the Fourier space. Both integrals lead to infinite values. The left-hand part of Fig. 3 depicts the integrand in the Fourier space; it does not vanish towards infinity. If all singular terms of the corresponding boundary integral equation are combined, then these singularities will disappear, as shown in the right-hand part of Fig. 3. The integrand is then totally regular and can be computed numerically without difficulties. This is possible for all singularities.

5. Conclusions

In contrast to the traditional approach, the Fourier BEM does not demand numerically pre-established tables for the fundamental solutions and interpolation during BEM analysis between these values. This enables a strategy that is faster, more direct, and less critical with respect to numerical errors. It is not advantageous when an analytical fundamental solution is known due to the higher effort required for the computation of the integrals. Then no tables have to be pre-evaluated and no interpolation is required. Nevertheless, the singular integrals (weak, strong, and hyper) occurring in standard BEM can also be computed in the Fourier space. It was shown that all non-integrable terms could be canceled, resulting in easy-to-handle standard integrals.

References

[1] Bonnet M. Boundary Integral Equation Methods for Solids and Fluids. New York: John Wiley and Sons, 1999.

[2] Duddeck F. Fourier BEM. Generalization of Boundary Element Methods by Fourier Transform. Lecture Notes in Applied Mechanics, Vol. 5. Berlin: Springer, 2002.

[3] Pomp A. The Boundary-Domain Integral Method for Elliptic Systems. Lecture Notes in Mathematics, Vol. 1683. Berlin: Springer, 1998.

[4] Andrä H, Schnack E: Integration of singular Galerkin-type boundary element integrals for 3D elasticity problems. Numerische Mathematik 1997;76:43–165.

[5] Gray LJ, Martha LF, Ingraffea AR. Hypersingular integrals in boundary element fracture analysis. Int J Numerical Meth Engng 1990;29:1135–1158.

[6] Guiggiani M. Hypersingular formulation for boundary stress evaluation. Engng Anal Boundary Elements 1994;13:169–179.