Superconvergence of linear functionals by discontinuous Galerkin approximations

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Abstract

We present a method for obtaining superconvergent approximations of linear functionals. We present an illustration of this idea in the framework of convection-diffusion equations. We use the approximation given by the discontinuous Galerkin method with polynomials of degree k. Instead of the classical order of convergence of $2k$, we prove that we can obtain an approximation of order $4k$. Numerical results that confirm this theoretical finding are presented.

Keywords: Convection–diffusion equation; Discontinuous Galerkin methods; Finite element methods; Functionals; Post-processing; Superconvergence

1. Introduction

We want to approximate the value of (u, g), where $(.,.)$ is the usual $L^2(\Omega)$-inner product, and u is the solution of the linear differential equation $Lu = f$ on the domain Ω. Following Pierce and Giles [1], we write

$$
\begin{aligned}
(u,g) &= (u_h,g) + (u-u_h,g) \\
&= (u_h,g) + (u-u_h,L^*v) \\
&= (u_h,g) + (L(u-u_h),v) \\
&= (u_h,g) + (L(u-u_h),v_h) + (L(u-u_h),v-v_h) \\
&= (u_h,g) + (f-Lu_h,v_h) + (L(u-u_h),v-v_h)
\end{aligned} \tag{1}
$$

where v is the solution of the adjoint problem $L^*v = g$ together with proper boundary conditions.

Pierce and Giles [1] noted that $J(u_h, v_h) := (u_h, g) + (f - Lu_h, v_h)$ is a better approximation to the functional value than just (u_h, g). Indeed,

$$|(u,g)-(u_h,g)| = |(L(u-u_h),v)| \le \|L(u-u_h)\|\,\|v\|$$

while

$$
\begin{aligned}
&|(u,g)-J(u_h,v_h)| \\
&\quad = |(L(u-u_h),v-v_h)| \le \|L(u-u_h)\|\,\|v-v_h\|
\end{aligned}
$$

Of course, these approximations are the same when $(f - Lu_h, v_h) = 0$, which typically is the case when u_h is a finite element solution. This is the so-called Galerkin orthogonality property. If u_h^* is a superconvergent post-processing of u_h, then (u_h^*, g) could be as good approximation as $J(u_h, v_h)$. However, as we have seen, $J(u_h^*, v_h^*)$ is an even better approximation to (u, g) given that there is no Galerkin orthogonality for u_h.

In this paper, we illustrate this in a simple framework of a convection–diffusion model problem. We take u_h to be the approximation given by the discontinuous Galerkin method and u_h^* a superconvergent post-processing of u_h. Thus, $(u, g) - (u_h, g)$ is $O(h^{2k})$ while $(u, g) - J(u_h^*, v_h^*)$ is $O(h^{4k})$.

2. The main result

Let us consider the following convection–diffusion problem:

$$
\begin{cases}
-\varepsilon u''(x) + cu'(x) = f(x) & \text{in } x \in (0,1) = \Omega \\
u = u_D & \text{in } x \in \{0,1\}
\end{cases} \tag{2}
$$

with $\varepsilon > 0$ and $c \ge 0$. Suppose we want to approximate the value $\int_0^1 u\,dx$.

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2.1. The scheme

As in Eq. (1), we can express the integral value as

$$\int_0^1 udx = (u,1) = (u_h^*,1) + (f - Lu_h^*, v_h^*) + (f - Lu_h^*, v_h^*)$$

where u_h^* and v_h^* are post-processed discontinuous Galerkin solutions.

We next define L^*:

$$\begin{aligned}(L\varphi, w) &= \int_0^1 (-\varepsilon\varphi'' + c\varphi')wdx \\ &= \int_0^1 (\varepsilon\varphi' - c\varphi)w'dx + (\varepsilon\varphi' - c\varphi)w\big|_0^1 \\ &= -\int_0^1 \varepsilon\varphi w''dx - \int_0^1 c\varphi w'dx + (-\varepsilon\varphi' + c\varphi)w\big|_0^1 + \varepsilon\varphi w'\big|_0^1 \\ &= \int_0^1 \varphi(-\varepsilon w'' - cw')dx + (-\varepsilon\varphi' + c\varphi)w\big|_0^1 + \varepsilon\varphi w'\big|_0^1 \\ &= (\varphi, -\varepsilon w'' - cw') + (-\varepsilon\varphi' + c\varphi)w\big|_0^1 + \varepsilon\varphi w'\big|_0^1\end{aligned}$$

The last term $\epsilon\varphi w'\big|_0^1$ is zero if φ satisfies the homogeneous boundary condition, and we can make $(-\varepsilon\varphi' + c\varphi)w\big|_0^1$ zero by imposing an homogeneous boundary condition on w. We note that $(u - u_h^*)(x) = 0$ at the boundary. Thus, $L^* = -\varepsilon\frac{d^2}{dx^2} - c\frac{d}{dx}$, and v is the solution of the adjoint problem

$$\begin{cases} L^*v = -\varepsilon v'' - cv' = -1 & \text{in } x \in \Omega, \\ v = 0 & \text{in } x \in \{0,1\} \end{cases} \tag{3}$$

2.2. Post-processing

In this subsection we will show how to construct u_h^* and v_h^*. We will first compute the discontinuous Galerkin approximations u_h and v_h. To that end, we employ the so-called local discontinuous Galerkin (LDG) method (see Castillo et al. [2]). We will solve an equivalent system to (2):

$$\begin{cases} q = \varepsilon u' & \text{in } x \in \Omega \\ -(q - cu)' & \text{in } x \in \Omega \\ u = u_D & \text{in } x \in \{0,1\} \end{cases}$$

Let $I_j = (x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}})$ for $j = 1, \ldots, N$ as a mesh for [0, 1], where $x_{\frac{1}{2}} = 0$ and $x_{N+\frac{1}{2}} = 1$. In this paper we will use the uniform mesh $|I_j| = h$ for all j for simplicity. We find the discontinuous Galerkin approximation u_h in the space $V_k(\Omega) = \{\varphi : \varphi \text{ is a polynomial of degree at most } k \text{ for } x \in I_j,\ j = 1, 2, \ldots, N\}$. Then u_h is determined by the relation

$$\begin{cases} \int_{I_j} q_h w dx + \varepsilon \int_{I_j} u_h w' dx - \varepsilon \hat{u}_{h_{j+\frac{1}{2}}} w^-_{j+\frac{1}{2}} + \varepsilon \hat{u}_{h_{j-\frac{1}{2}}} w^+_{j-\frac{1}{2}} = 0 \\ \int_{I_j} (q_h - cu_h)\varphi' dx - (\hat{q}_h - c\hat{u}_h)_{j+\frac{1}{2}} \varphi^-_{j+\frac{1}{2}} + (\hat{q}_h - c\hat{u}_h)_{j-\frac{1}{2}} \varphi^+_{j-\frac{1}{2}} = \int_{I_j} \int \varphi \end{cases}$$

for all w, $\varphi \in V_k(\Omega)$, and by the numerical flux

$$\begin{cases} \hat{u}_{h_{j+\frac{1}{2}}} = u_D(0) & j = 0 \\ \hat{u}_{h_{j+\frac{1}{2}}} = u^-_{h_{j+\frac{1}{2}}} & j = 1, 2, \ldots, N-1 \\ \hat{u}_{h_{j+\frac{1}{2}}} = u_D(1) & j = N \\ \hat{q}_{h_{j+\frac{1}{2}}} = q^+_{h_{j+\frac{1}{2}}} & j = 0, 1, \ldots, N-1 \\ \hat{q}_{h_{j+\frac{1}{2}}} = q^-_{h_{N+\frac{1}{2}}} + \frac{\varepsilon}{h}(u_D(1) - u^-_{h_{N+\frac{1}{2}}}) & j = N \end{cases}$$

Such numerical flux is called 'upwinding'. Celiker and Cockburn [3] proved that the above scheme had superconvergence in numerical fluxes of order $2k + 1$ at the nodes. Using this result, we may do post-processing to construct a better approximate solution u_h^*. Here we used the Lagrange interpolation

$$u_h^*(x) = \sum_{j=0}^{2k} \hat{u}_h(x_j) \prod_{l=0, l\neq j}^{2k+1} \frac{x - x_l}{x_j - x_l} \tag{4}$$

The post-processed solution v_h^* of the dual problem defined by Eqs. (3) is constructed in a similar way.

Theorem 1 [4] Let u_h^* and v_h^* be defined as in Eq. (4), interpolating the corresponding numerical fluxes. Then $J = (u_h^*, 1) + (L(u - u_h^*), v_h^*)$ converges to $\int_0^1 udx$ with order $4k$.

Numerical results obtained for $\in = 1$, $c = \frac{1}{2}$, $u_0 = 0$, $u_1 = \sin(1)$, and $f(x) = \sin(x) + \frac{1}{2}\cos(x)$ are shown in Table 1. Figure 1 shows the log of remaining error versus the log of number of elements for $k = 1, 2, \ldots, 5$. For each polynomial degree one sees that the error superconverges with order at least $4k$.

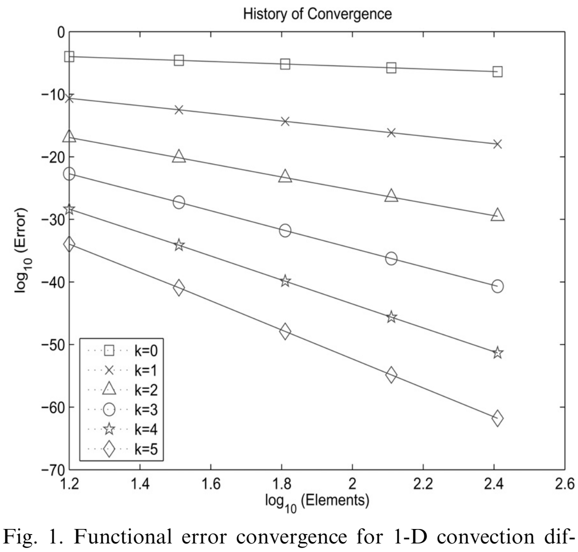

Fig. 1. Functional error convergence for 1-D convection diffusion equation.

Table 1: History of convergence

	$k = 0$		$k = 1$		$k = 2$		$k = 3$	
Elements	Error	Rate	Error	Rate	Error	Rate	Error	Rate
8	4.13E-04	–	1.83E-09	–	2.47E-14	–	6.98E-19	–
16	1.03E-04	2	2.26E-11	6.34	1.23E-17	10.97	1.94E-23	15.13
32	2.58E-05	2	3.13E-13	6.18	7.05E-21	10.77	5.51E-28	15.1
64	6.45E-06	2	4.60E-15	6.09	4.72E-24	10.54	1.68E-32	15.01
128	1.61E-06	2	6.97E-17	6.04	3.63E-27	10.35	5.54E-37	14.88
256	4.04E-07	2	1.07E-18	6.02	3.09E-30	10.2	2.03E-41	14.74

	$k = 4$		$k = 5$	
Elements	Error	Rate	Error	Rate
16	4.34E-29	–	1.09E-34	–
32	7.40E-35	19.16	1.16E-41	23.17
64	1.31E-40	19.11	1.26E-48	23.13
128	2.40E-46	19.05	1.42E-55	23.08
256	4.57E-52	19	1.64E-62	23.04

3. Conclusions

In this paper, we have shown that functionals associated with the solution of a convection-diffusion equation can be approximated with accuracy of $O(h^{4k})$. It is contrasted with the usual order of convergence, which is $O(h^{2k})$. The extension of this work to more general settings and to the multi-dimensional case is the subject of our future research.

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