Meshless domain decomposition schemes for nonlinear elliptic PDEs

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Abstract

In this paper, we present a radial basis function based meshless method that employs Schwarz domain decomposition techniques and Newton iteration for solving nonlinear elliptic partial differential equations (PDEs). Numerical results are presented for a model nonlinear Poisson problem. We find that the domain decomposition algorithms give accurate results at a much lower cost. In addition, these algorithms are straightforward to parallelize.

Keywords: Nonlinear PDEs; Schwarz domain decomposition; Radial basis functions; Newton iteration

1. Introduction

In recent years, the application of radial basis functions (RBFs) to solve partial differential equations (PDEs) has become very popular. Kansa [1,2], introduced the unsymmetric RBF collocation technique motivated by advances in function approximation theory. This method is a truly meshless scheme since it makes use of only a scattered set of collocation points in the domain and no connectivity information is required. The most commonly used globally supported RBFs are multiquadrics (MQ) [$\sqrt{r^2+\sigma_2}$], thin plate splines [$r^4 \log r$, $r^8 \log r$], and Gaussians [$\exp(-r^2/\sigma^2)$]. Here, r denotes the Euclidean norm and σ is a shape parameter that controls the region of influence of the RBF.

The solution of nonlinear PDEs using RBFs has been studied previously [3–6]. Fasshauer solved nonlinear elliptic problems using a multilevel Newton iteration method coupled with RBFs [5]. He found that the MQ globally supported RBF produced better results compared with finite elements (FE) and locally supported RBFs, even when a coarse set of collocation points are used. However, the RBF approach can be computationally expensive for a large number of collocation points [5].

In the case of the RBF interpolation technique, Schaback's uncertainty principle [7] points out that the coefficient matrices become progressively ill-conditioned as the accuracy of the RBF interpolant improves. This principle applies even for the solution of PDEs. Kansa and Hon [8] have suggested some techniques to circumvent the ill-conditioning problem, such as using truncated RBFs, preconditioning, and domain decomposition methods (DDMs). For some recent work on DDMs using RBFs, the reader is referred to Zhou et al. [9], Li and Hon [10], and Chinchapatnam et al. [11]. These papers illustrate the efficiency of DDMs for linear elliptic and time-dependent PDEs. In this paper, we extend our earlier work [11] to apply the RBF-based overlapping Schwarz DDMs coupled with operator Newton iteration for solving nonlinear elliptic problems. Numerical studies show that DDMs give results at a much lower computational cost compared with the standard RBF collocation technique while incurring a slight loss of accuracy.

In Section 2, we present the Newton iteration method coupled with the standard unsymmetric collocation method for the solution of nonlinear PDEs. In the remainder of the paper, we refer to this method as the standard Newton iteration RBF scheme. In Section 3, we introduce Schwarz domain decomposition methods coupled with RBFs. In Section 4, numerical studies are presented for a model problem to illustrate the efficiency of the proposed domain decomposition schemes.

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2. RBF collocation method for nonlinear PDEs

In the present work, we consider nonlinear elliptic PDEs of the form

$$\mathcal{L}u(\boldsymbol{x}) = f(\boldsymbol{x}),\ \boldsymbol{x} \in \Omega \subset \mathbb{R}^d$$
$$\mathcal{B}u(\boldsymbol{x}) = g(\boldsymbol{x}),\ \boldsymbol{x} \in \partial\Omega \subset \mathbb{R}^d \tag{1}$$

where $\mathcal{L}$ is a nonlinear differential operator and $\mathcal{B}$ is a boundary operator (Dirichlet, Neumann or mixed). Ω represents a bounded physical domain and $\partial\Omega$ represents its boundary. $f(\boldsymbol{x})$ and $g(\boldsymbol{x})$ are some prescribed functions.

Eq. (1) can be solved using the Newton iteration method. In this method, one starts with an initial solution $u^0(\boldsymbol{x})$ and then obtains a sequence $\{u^i\}$ of approximate solutions using the Newton iteration formula. Let $\mathcal{L}^{i-1}$ be the linearized version of the nonlinear differential operator $\mathcal{L}$ at iteration $i-1$; then the update at $i-1$ is the solution of the following linear operator problem:

$$\mathcal{L}^{i-1}v(\boldsymbol{x}) = f(\boldsymbol{x}) - \mathcal{L}u^{i-1}(\boldsymbol{x}),\ \boldsymbol{x} \in \Omega$$
$$\mathcal{B}v(\boldsymbol{x}) = g(\boldsymbol{x}) - \mathcal{B}u^{i-1}(\boldsymbol{x}),\ \boldsymbol{x} \in \partial\Omega \tag{2}$$

Once $v(\boldsymbol{x})$ is obtained by solving Eq. (2), then the new estimate at iteration i is given by

$$u^i(\boldsymbol{x}) = u^{i-1}(\boldsymbol{x}) + v(\boldsymbol{x}) \tag{3}$$

The RBF collocation method is utilized to solve Eq. (2). At each iteration, the Newton update $v(\boldsymbol{x})$ is assumed to be given by

$$v(\boldsymbol{x}) = \sum_{j=1}^{N} \lambda_j \phi\left(\left\|\boldsymbol{x} - \boldsymbol{c_j}\right\|\right) \tag{4}$$

where $\phi(\|\boldsymbol{x} - \boldsymbol{c_j}\|)$: $\mathbb{R}^d \to \mathbb{R}$ is an RBF centered on $\boldsymbol{c_j} \in \mathbb{R}^d$. These centers are chosen from the set $\mathcal{C} = \{(\boldsymbol{c}_i)|_{i=1,n_d} \in \Omega,\ (\boldsymbol{c}_i)|_{i=n_d+1,n_d+n_b} \in \partial\Omega\}$, where n_d and n_b denote the number of centers inside the domain and on the boundary, respectively, and $N = n_d + n_b$. For the sake of simplicity, we assume that the RBF centers coincide with the collocation points.

Substituting Eq. (4) into Eq (2) and collocating on each of the points in $\mathcal{C}$, we obtain the following linear algebraic system of equations

$$\sum_{j=1}^{N} \lambda_j \mathcal{L}^{i-1}\phi\left(\left\|\boldsymbol{x_i} - \boldsymbol{c_j}\right\|\right) = F(\boldsymbol{x_i}), \qquad i = 1, 2, \ldots, n_d$$
$$\sum_{j=1}^{N} \lambda_j \mathcal{B}\phi\left(\left\|\boldsymbol{x_i} - \boldsymbol{c_j}\right\|\right) = G(\boldsymbol{x_i}), \quad i = n_d + 1, n_d + 2, \ldots, N \tag{5}$$

which can be solved for the unknown coefficients λ_j, $j = 1, 2, \ldots, N$.

Note that the coefficient matrix of the above system of equations changes at every Newton iteration and thus the computational cost of the RBF method is increased, as the matrix needs to be decomposed at each iteration. Also, the matrix formed suffers from ill-conditioning at the optimal values of the shape parameter when multiquadric RBFs are used [7].

In the present paper, we extend the standard Newton iteration RBF method by employing domain decomposition techniques to efficiently solve Eq. (2). Using the DDMs, it is found that the coefficient matrices in each subdomain have better condition numbers and the whole process tends to become much faster compared with the standard Newton iteration RBF scheme. It is worth noting that the Schwarz domain decomposition schemes presented in this paper can also be coupled with other techniques, such as the fixed point iteration method.

3. Domain decomposition scheme

Domain decomposition methods using RBFs may be applied to solve the Newton update problem in Eq. (2). For simplicity of presentation, we illustrate a Schwarz additive domain decomposition for the case when the domain Ω is partitioned into two overlapping subdomains Ω_1 and Ω_2. Also, let Γ_k denote the part of boundary of Ω_k that is interior to Ω (artificial boundary) and $\partial\Omega_k \backslash \Gamma_k$ denote the natural boundaries of Ω_k subdomain ($k = 1, 2$). A schematic diagram of the domain is shown in Fig. 1.

At each outer Newton update iteration i, the subdomain problems can be written as

$$\mathcal{L}^{i-1}v_1^{i,j} = f - \mathcal{L}u_1^{i-1} \text{in} \quad \Omega_1; \qquad \mathcal{L}^{i-1}v_2^{i,j} = f - \mathcal{L}u_2^{i-1} \text{ in} \quad \Omega_2$$
$$\mathcal{B}v_1^{i,j} = g - \mathcal{B}u_1^{i-1} \text{on} \quad \partial\Omega_1 \backslash \Gamma_1; \ \mathcal{B}u_2^{i,j} = g - \mathcal{B}u_2^{i-1} \text{ on} \quad \partial\Omega_2 \backslash \Gamma_2$$
$$\mathcal{S}v_1^{i,j} = v_2^{i,j-1} \text{on} \quad \Gamma_1; \qquad \mathcal{S}v_2^{i,j} = v_1^{i,j-1} \text{ on} \quad \Gamma_2 \tag{6}$$

where $\mathcal{S}$ denotes the artificial boundary operator, which can be a Dirichlet, Neumann, or mixed operator, and j denotes the iteration number of the inner Schwarz iterations.

Eq. (6) can be solved using the RBF collocation method presented in Section 2. In other words, $v_1(\boldsymbol{x})$ and $v_2(\boldsymbol{x})$ are approximated using expansions of the form given in Eq. (4). For the Schwarz additive algorithm, the values of $v(\boldsymbol{x})$ on the artificial boundaries are updated after solving the operator problem for each subdomain Ω_1 and Ω_2. On the overlapping regions, the value of the Newton update $v(\boldsymbol{x})$ is taken to be the average of the values of $v_1(\boldsymbol{x})$ and $v_2(\boldsymbol{x})$ in order to enforce continuity

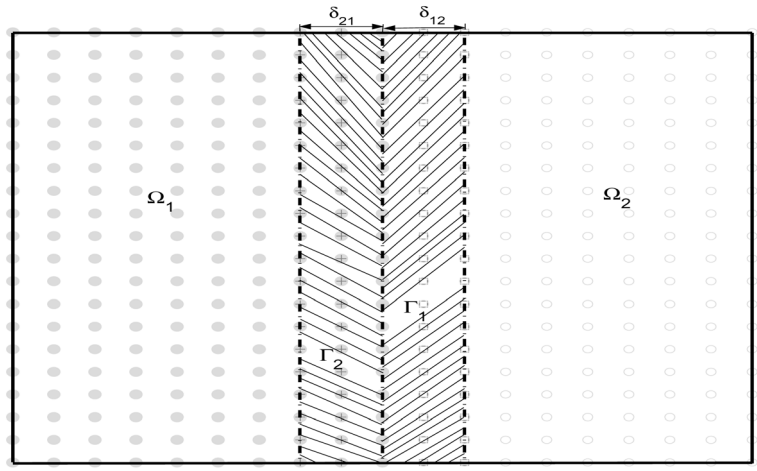

Fig. 1. Point distributions in the Schwarz subdomains.

across the artificial boundary. In the Schwarz multiplicative version of the algorithm, the value of $v(\boldsymbol{x})$ is updated in a sequential fashion, i.e. the value $v_1(\boldsymbol{x})$ obtained by solving the operator problem on Ω_1 is substituted into the right-hand side of the operator problem for Ω_2. The Schwarz iterations continue until the difference in the values of $v(\boldsymbol{x})$ for subsequent iterations is less than some fixed value ε. Once Eq. (6) is solved to obtain the value of $v(\boldsymbol{x})$, it can be added to the previous estimate $u^{i-1}(\boldsymbol{x})$ to obtain the new estimate. Likewise, the Newton iterations continue until the solution converges.

4. Numerical studies

To illustrate the performance of the proposed domain decomposition algorithms, we use the following two-dimensional (2D) nonlinear PDE taken from Fasshauer [5]:

$$\begin{aligned} -\varepsilon^2\nabla^2 u(\boldsymbol{x}) - u(\boldsymbol{x}) + u(\boldsymbol{x})^3 &= f(\boldsymbol{x}), \quad \boldsymbol{x} \in \Omega = (0,1)^2 \\ u(\boldsymbol{x}) &= 0 \qquad\qquad \boldsymbol{x} \in \partial\Omega \end{aligned} \tag{7}$$

The function $f(\boldsymbol{x})$ is chosen so that Eq. (7) has an analytic solution of the form $u(x, y) = \psi(x)\psi(y)$ with $\psi(t) = 1 + e^{-1/\varepsilon} - e^{-t/\varepsilon} - e^{(t-1)/\varepsilon}$. Here, (x, y) denote the Cartesian coordinates of $\boldsymbol{x} \in \mathbb{R}^2$, and the parameter ε determines the size of the boundary layers near the edges of the domain Ω. We use $\varepsilon = 0.1$ in the presented studies.

The accuracy of the results is estimated using the following error norm calculated on a very fine uniform mesh (50×50 collocation points),

$$e = \sqrt{\frac{1}{m}\sum_{j=1}^{m}\left|u_{\text{analytic}}(\boldsymbol{x}_j) - u_{\text{numerical}}(\boldsymbol{x}_j)\right|^2},$$

where m is the number of points in the fine mesh

For the Schwarz additive case, the subdomain problem to be solved is given by

$$\begin{aligned} -\varepsilon^2\nabla^2 v_k^{i,j} + [3(u^{i-1})^2 - 1]v_k^{i,j} &= f + \varepsilon^2\nabla^2 u_k^{i-1} + u_k^{i-1} - (u_k^{i-1})^3, \ \text{in } \Omega_k \\ v_k^{i,j} &= 0, \qquad \text{on } \partial\Omega_k\backslash\Gamma_k \\ v_k^{i,j} &= v_l^{i,j-1}, \qquad \text{on } \Gamma_k \end{aligned} \tag{8}$$

where i represents the outer Newton iteration number and j represents the iteration number of the inner Schwarz iterations. The inner iterations are terminated when $\left\|v_k^{i,j} - v_k^{i,j-1}\right\|_2 < 1.0\text{E}-03, \quad \text{k} = 1, 2, \ldots, \text{N}_\text{e}$, where N_e denotes the number of subdomains.

We use the MQ RBF for the numerical studies. The shape parameter was chosen from the set $[\frac{2}{\sqrt{N}}, \infty)$, such that the residual error calculated for the linear problem at each iteration is minimum. A comparison of the results obtained using the standard Newton iteration RBF scheme using 1600 points with the analytical solution is shown in Fig. 2.

In Table 1, we present the results of the Schwarz domain decomposition schemes for the case when the number of subdomains (N_e) is equal to two along with the standard Newton iteration RBF scheme (i.e. $N_e = 1$). The number in brackets gives the value of the shape parameter used. The computational cost is assumed to be proportional to the CPU time. All the calculations were performed on an AMD Athlon machine with an MP 2600+ processor. For the first three entries in the

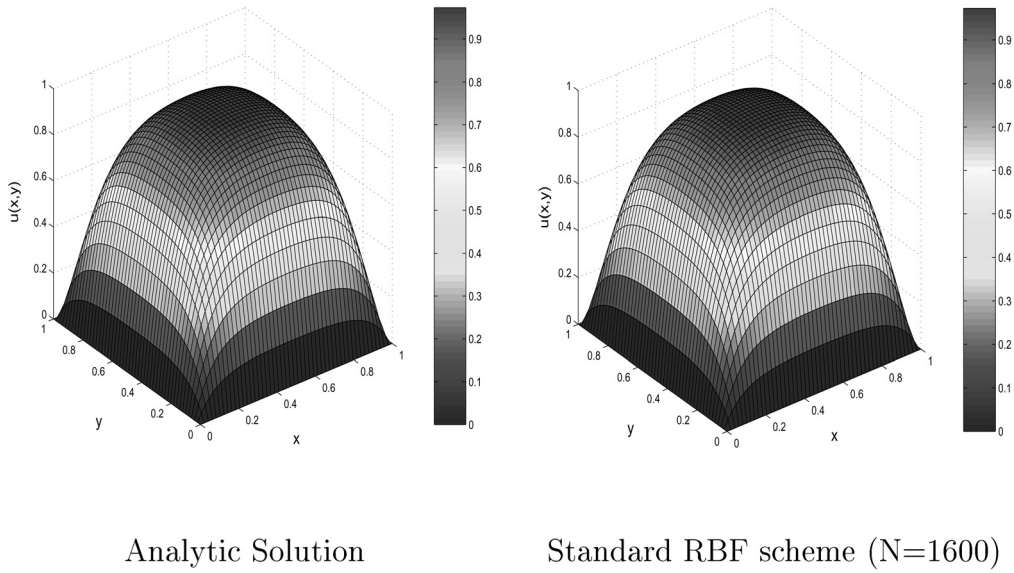

Fig. 2. Comparison of RBF–Newton iteration scheme result with analytical solution.

Table 1
Domain decomposition scheme results using MQ RBF

	$N_e = 1$		$N_e = 2$			
	Standard Newton		Additive Schwarz		Multiplicative Schwarz	
N	L_2 error (σ)	CPU time (s)	L_2 error (σ)	CPU time (s)	L_2 Error (σ)	CPU time (s)
16	$1.92E-01$ (2.5)	0	–	–	–	–
36	$7.76E-02$ (2.5)	0	–	–	–	–
100	$1.45E-02$ (1.1)	1	–	–	–	–
400	$8.62E-04$ (0.5)	8	$4.83E-03$ (0.5)	3	$3.92E-03$ (0.5)	2
900	$2.90E-04$ (0.3)	17	$2.96E-03$ (0.3)	7	$2.40E-03$ (0.3)	10
1600	$1.90E-04$ (0.2)	46	$1.23E-03$ (0.2)	22	$1.0E-03$ (0.2)	28

table, the results for DDMs are not given, as the computational cost incurred by the DDM method is insignificant. From the results, it can be seen that for $N_e = 2$, DDMs are much faster than the single-domain Newton iteration method, with a slight loss of accuracy.

Next, we investigate the influence of number of subdomains on the accuracy and computational cost. The total number of collocation points was fixed at 1,600 and the physical domain was divided into two, four, and eight subdomains. The results are presented in Fig. 3. From this figure, it can be seen that as the number of subdomains increases, the computational cost decreases. However, the accuracy of the method also suffers.

5. Conclusion

Meshless overlapping Schwarz additive and multiplicative DDMs coupled with Newton iterations are presented for solving nonlinear elliptic PDEs. The performance of these techniques was compared with that of the standard RBF method coupled with Newton iteration by solving a 2D nonlinear problem. The DDMs were found to be much faster than the standard Newton iteration RBF scheme. However, as the number of subdomains increases, the computational cost decreases at the expense of accuracy. Also, it is worth noting that using piecewise linear finite elements, 16,384 nodes are needed to obtain an accuracy of $\mathcal{O}(10^{-3})$ [5]. In comparison, the RBF studies achieve similar accuracy using only 400 collocation points. Finally, the presented algorithms are suitable for parallel implementation.

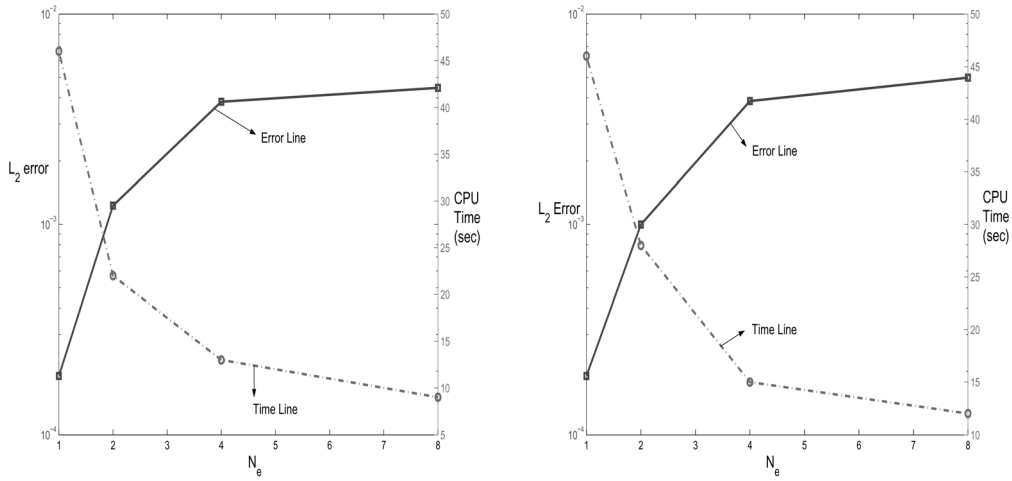

Fig. 3. Influence of number of domains.

References

[1] Kansa EJ. Multiquadrics: a scattered data approximation scheme with applications to computation fluid dynamics: I. Surface approximations and partial derivatives estimates: Comput Math Appl 1990;19:127–145.

[2] Kansa EJ. Multiquadrics: a scattered data approximation scheme with applications to computation fluid dynamics: II. Solution to parabolic, hyperbolic and elliptic partial differential equations. Comput Math Appl 1990;19:147–161.

[3] Hon YC, Mao XZ. An efficient numerical scheme for Burger's equation. Appl Math Comput 1998;95:37–50.

[4] Wong ASM, Hon YC, Li TS, Cheng SL, Kansa EJ. Multizone decomposition for simulation of time-dependent problems using the multiquadric scheme. Comput Math Appl 1999;37:23–43.

[5] Fasshauer GE. Newton iteration with multiquadrics for the solution of nonlinear PDEs. Comput Math Appl 2002;43:423–438.

[6] Fedoseyev AI, Friedman MJ, Kansa EJ. Continuation for nonlinear elliptic partial differential equations discretized by the multiquadric method. Int J Bifur Chaos 2000;2:481–492.

[7] Schaback R. Error estimates and condition numbers for radial basis interpolation. Adv Comput Math 1995;3:251–264.

[8] Kansa EJ, Hon YC. Circumventing the ill-conditioning problem with multiquadric radial basis functions. Comput Math Appl 2000;39:123–137.

[9] Zhou X, Hon YC, Li J. Overlapping domain decomposition method by radial basis functions. Appl Number Math 2003;44:241–255.

[10] Li J, Hon YC. Domain decomposition for radial basis meshless methods. Numer Meth Partial Differential Equations 2004;20:450–462.

[11] Chinchapatnam PP, Djidjeli K, Nair PB. Domain decomposition for time-dependent problems using radial based meshless methods. Numer Meth Partial Differential Equations 2004.