A return mapping algorithm for isotropic and anisotropic large deformations

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Abstract

A return mapping (fully implicit) algorithm suitable for general isotropic and anisotropic hyperelastoplasticity is developed. The algorithm is cast in the framework of multiplicative decomposition and has no restriction to hyperelastic isotropy, isotropic flow function, or isotropic yield function. As such, it can be applied to both isotropic and anisotropic large-deformation problems. Numerical examples illustrating good performance are shown.

Keywords: Large deformation; Return mapping; Hyperelastoplasticity; Anisotropy

1. Introduction

The return mapping family of algorithms has had a great success within small-deformation theories [1,2]. The use of algorithmic tangent stiffness in conjunction with global Newton iterative scheme can, in theory, provide for a quadratic convergence rate for equilibrium iterations. We mention a few related papers chosen from a large number of works on this subject [4–7]. In Simo [8,9], the return mapping algorithm was extended to the large-deformation regime and then used successfully by other authors [10–16]. However, the extensions are strictly restricted to isotropic problems. These restrictions include at least three aspects. First, the hyperelastic strain energy function must be an isotropic function of strains in the intermediate configuration and thus is not consistent with anisotropic hyperelasticity. Second, the approach is valid only on the condition that the flow rule is an isotropic function of the stress tensor. Third, the approach assumes that the yield function is an isotropic function of stresses in the current configuration, in order to retain frame invariance. The above three assumptions are apparently violated for initially anisotropic models, models with kinematic hardening, and models with induced anisotropy.

It should be noted that very few researchers have addressed the issue of large-deformation hyperelastoplastic computational formulations for anisotropic materials. We mention an algorithm by Eterovic and Bathe [17], which is based on an additive split of logarithmic stress and strain measures (elastic and hyperelastoplastic). They have also explored the use of a more general approximation of deformation tensors that can support both isotropic and anisotropic material models. More recently, Papadopoulos and Lu [18] developed a general framework for finite deformation elastoplasticity that is based on the early work of Green and Naghdi [19]. Developments include provisions for non-collinearity of principal stress and strain measures. The framework was tested using von Mises-type yield criteria with translational kinematic hardening, which retain isotropy of yield and plastic potential functions. Somewhat similar developments were reported by Miehe et al. [20,21]. They used initial anisotropy in hyperelastic models and initially anisotropic criterion by Hill [22] to successfully simulate various problems. However, it was not clear if, and how, the non-collinearity of principal stress and strain measures evolve and how much it influences the results.

In this paper, we present hyperelastoplastic constitutive relations and the fully implicit return mapping algorithm that can handle general anisotropic hyperelastoplastic material models subjected to monotonic and/or cyclic loading. We present some numerical examples to illustrate the good performance of the algorithm.

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2. Hyperelastic-plastic constitutive relations

The constitutive relations for large-deformation inelastic problems include five parts: multiplicative decomposition of the gradient of deformation, hyperelastic relations, yield function, flow rule, and hardening/softening laws.

The multiplicative decomposition of the deformation gradient[1] $F_{ij} = F^e_{ik} F^p_{kj}$ proposed by Bilby et al. [23], Kröner [24], Lee and Liu [25], and Lee [26], plays a fundamental role in the hyperelastic-plastic constitutive models. This assumes an intermediate configuration between the initial and current configuration as $\Omega_0 \xrightarrow{F^p} \bar{\Omega} \xrightarrow{F^e} \Omega$. Although the decomposition of the deformation gradient is multiplicative, it results directly in the additive velocity gradient decomposition in the intermediate configuration, which is similar to the infinitesimal counterpart:

$$\bar{d}_{ij} = \dot{\bar{E}}^e_{ij} + \bar{d}^p_{ij} \tag{1}$$

where $\bar{d}_{ij} = Sym(\dot{F}_{ik} F^{-1}_{kj})$ is the total velocity gradient in the intermediate configuration, $\bar{E}^e_{ij} = (\bar{C}^e_{ij} - \delta_{ij})/2$ is the elastic Green strain in the intermediate configuration, the Kronecker delta δ_{ij} is the unit tensor, and $\bar{C}^e_{ij} = F^e_{ki} F^e_{kj}$, the plastic velocity gradient in the intermediate configuration is given as $\bar{d}^p_{ij} = Sym(\bar{C}^e_{ik} \bar{L}^p_{kj})$, with $\bar{L}^p_{ij} = F^{p-1}_{ik} F^p_{kj}$.

The hyperelasticity is usually defined in terms of the second Piola–Kirchhoff stress in the intermediate configuration $\bar{S}_{ij}$:

$$\bar{S}_{ij} = \frac{\partial \Psi}{\partial \bar{E}^e_{ij}} \quad ; \quad \dot{\bar{S}}_{ij} = \bar{\mathcal{L}}_{ijkl} \dot{\bar{E}}^e_{kl} \quad ; \quad \bar{\mathcal{L}}_{ijkl} = \frac{\partial^2 \Psi}{\partial \bar{E}^e_{ij} \partial \bar{E}^e_{kl}} \tag{2}$$

where Ψ is the hyperelastic potential function and *Sym* means the symmetric part. We transfer the hyperelasticity in terms of Mandel stress in the intermediate configuration $\bar{T}_{ij}$, since this is convenient for defining the flow rules:

$$\bar{T}_{ij} = \bar{C}^e_{ij} \bar{S}_{ij} \quad ; \quad \dot{\bar{T}}_{ij} = \bar{\mathcal{L}}^M_{ijmn} \dot{\bar{E}}^e_{mn} \quad ; \quad \bar{\mathcal{L}}^M_{ijmn} = \delta_{im} \bar{S}_{jn} + \delta_{in} \bar{S}_{jm} + \bar{C}^e_{ik} \bar{\mathcal{L}}_{kjmn} \tag{3}$$

The yield function in terms of Mandel stress $\bar{T}_{ij}$ and stress-like internal isotropic variables q, and/or kinematic variable a_{ij}, is then given in the following form:

$$\mathcal{F} = \mathcal{F}(\bar{T}_{ij}, q, a_{ij}) \tag{4}$$

The flow rules for strain-like internal variables are not necessarily associated with the yield function. They are assumed to be:

$$\bar{L}^p_{ij} = \dot{\lambda} \bar{M}_{ij}(\bar{T}_{ij}, q, a_{ij}); \quad \text{or} \quad \bar{d}^p_{ij} = \dot{\lambda} \bar{M}^C_{ij} \quad ; \quad \bar{M}^C_{ij} = Sym\left(\bar{C}^e_{ik} \bar{M}_{kj}\right) \tag{5}$$

$$\dot{\xi} = -\dot{\lambda} n^i(\bar{T}_{ij}, q, a_{ij}) \quad ; \quad \dot{\eta}_{ij} = -\dot{\lambda} n^k_{ij}(\bar{T}_{ij}, q, a_{ij}) \tag{6}$$

where λ is the consistent plastic multiplier and ξ and η_{ij} are, respectively, the conjugate strain-like internal variables of q and a_{ij}.

The isotropic and kinematic hardening laws are given as:

$$\dot{q} = K\dot{\xi} \quad ; \quad \dot{a}_{ij} = \mathcal{H}_{ijkl} \dot{\eta}_{kl} \tag{7}$$

where K and $\mathcal{H}_{ijkl}$ are the isotropic scalar modulus and kinematic tensorial modulus, respectively. When K and $\mathcal{H}_{ijkl}$ are constant, one obtains the linear isotropic or kinematic hardening law.

Finally, similar to the small-deformation elastoplastic theory, the Kuhn–Tucker [27] conditions should be satisfied:

$$\dot{\lambda} \geq 0 \quad ; \quad \mathcal{F}(\bar{T}_{ij}, q, a_{ij}) \leq 0 \quad ; \quad \dot{\lambda} \mathcal{F}(\bar{T}_{ij}, q, a_{ij}) = 0 \tag{8}$$

3. Return mapping algorithm

By employing time integration of Eq. (5), and then using Taylor's series expansion and neglecting the higher-order terms of $\Delta\lambda$, we obtain the basic return mapping relation:

$$^{n+1}\bar{E}^e_{ij} = {}^{n+1}\bar{E}^{e,trial}_{ij} - \Delta\lambda \bar{M}^{C,trial}_{ij} \quad ; \quad \bar{M}^{C,trial}_{ij} = Sym\left({}^{n+1}\bar{C}^{e,trial}_{ik}\, {}^{n+1}\bar{M}_{kj}\right) \tag{9}$$

3.1. Initialization of variables

The known variables inherited from the previous time step are $\left\{ {}^n F^p_{ij}, {}^n\xi, {}^n q, {}^n\eta_{ij}, {}^n a_{ij} \right\}$. Given a trial $^{n+1}F^{trial}_{ij}$ during the time step $[{}^n t, {}^{n+1} t]$ (the iterative steps are denoted by right upper numbers in the brackets), at the iteration step 0, set $\Delta\lambda^{(0)} = 0$, $^{n+1}\xi^{(0)} = {}^n\xi$, $^{n+1}q^{(0)} = {}^n q$, $^{n+1}\eta^{(0)}_{ij} = {}^n\eta_{ij}$, $^{n+1}a^{(0)}_{ij} = {}^n a_{ij}$ and $^{n+1}F^{e,(0)}_{ij} = {}^{n+1}F^{trial}_{ik} ({}^n F^p_{kj})^{-1}$.

3.2. Check for convergence at iteration step (k)

If $|^{n+1}\mathcal{F}^{(k)}(^{n+1}\bar{T}^{(k)}_{ij},\ ^{n+1}q^{(k)},\ ^{n+1}\eta^{(k)}_{ij})| \leq \epsilon_1$ and $\|^{n+1}\tilde{\mathbf{r}}^{(k)}\| \leq \epsilon_2$, where ϵ_1, ϵ_2 are certain small (relative) tolerance values and $\tilde{\mathbf{r}}$ is defined in Eq. (13), exit the procedure; else, proceed to the next step (from now on the left upper subscripts $(n+1)$ are omitted for brevity).

3.3. Calculate change in consistency parameter $\Delta^2\lambda^{(k)}$

$$\Delta^2\lambda^{(k)} = \frac{\mathcal{F}^{(k)} - \left(\tilde{\mathbf{f}}_A^{(k)}\right)^T \mathbb{C}_{AB}^{(k)} \tilde{\mathbf{r}}_B^{(k)}}{\left(\tilde{\mathbf{f}}_A^{(k)}\right)^T \mathbb{C}_{AB}^{(k)} \tilde{\mathbf{m}}_B^{(k)}} \tag{10}$$

where $\mathbb{C}_{AB}$, $\tilde{\mathbf{f}}_A$, $\tilde{\mathbf{r}}_A$, and $\tilde{\mathbf{m}}_A$ are generalized matrices and vectors in which the element components can be mixed tensors and/or scalar. The indices with capital letters (A) and (B) may be, for example, (ij) for rank-2 tensors and/or nothing for scalars.

$$\mathbb{C}_{AB}^{(k)} = \begin{bmatrix} \delta_{ik}\delta_{jl} + \Delta\lambda \frac{\partial \bar{M}_{ij}^{C,(k)}}{\partial \bar{T}_{mn}} \bar{\mathcal{L}}_{mnkl}^{M,(k)} & \Delta\lambda \frac{\partial \bar{M}_{kl}^{C,(k)}}{\partial q} K^{(k)} & \Delta\lambda \frac{\partial \bar{M}_{ij}^{C,(k)}}{\partial a_{mn}} \mathcal{H}_{mnkl}^{(k)} \\ \Delta\lambda \frac{\partial n^{i,(k)}}{\partial \bar{T}_{ij}} \bar{\mathcal{L}}_{ijkl}^{M,(k)} & 1 + \Delta\lambda \frac{\partial n^{i,(k)}}{\partial q} K^{(k)} & \Delta\lambda \frac{\partial n^{i,(k)}}{\partial a_{ij}} \mathcal{H}_{ijkl}^{(k)} \\ \Delta\lambda \frac{\partial n_{ij}^{k,(k)}}{\partial \bar{T}_{mn}} \bar{\mathcal{L}}_{mnkl}^{M,(k)} & \Delta\lambda \frac{\partial n_{kl}^{k,(k)}}{\partial \bar{T}_{mn}} K^{(k)} & \delta_{ik}\delta_{jl} + \Delta\lambda \frac{\partial n_{ij}^{k,(k)}}{\partial a_{mn}} \mathcal{H}_{mnkl}^{(k)} \end{bmatrix}^{-1} \tag{11}$$

$$\tilde{\mathbf{f}}_A^{(k)} = \left\{ \frac{\partial \mathcal{F}^{(k)}}{\partial \bar{T}_{ij}} \mathcal{L}_{ijkl}^{M,(k)} \quad \frac{\partial \mathcal{F}^{(k)}}{\partial q} K^{(k)} \quad \frac{\partial \mathcal{F}^{(k)}}{\partial a_{ij}} \mathcal{H}_{ijkl}^{(k)} \right\}^T \tag{12}$$

$$\tilde{\mathbf{r}}_A^{(k)} = \left\{ \bar{R}_{mn}^{(k)} \; r^{i,(k)} \; r_{ij}^{k,(k)} \right\}^T \tag{13}$$

$$\tilde{\mathbf{m}}_A^{(k)} = \left\{ \bar{M}_{mn}^{c,(k)} \; n^{i,(k)} \; n^{k,(k)} \right\}^T \tag{14}$$

$$\Delta\tilde{\mathbf{u}}_A = \left\{ \Delta\bar{E}_{kl}^{e,(k)} \; \Delta\xi^{(k)} \; \Delta\eta_{kl}^{(k)} \right\}^T = -\mathbb{C}_{AB}^{(k)}\left(\tilde{\mathbf{r}}_B^{(k)} + \Delta^2\lambda^{(k)}\tilde{\mathbf{m}}_B^{(k)}\right) \tag{15}$$

3.4. Update variables for next iteration step

$$\Delta\lambda^{(k+1)} = \Delta\lambda^{(k)} + \Delta^2\lambda^{(k)} \tag{16}$$

$$\bar{E}_{ij}^{e,(k+1)} = \bar{E}_{ij}^{e,(k)} + \Delta\bar{E}_{ij}^{e,(k)}; \; \xi^{(k+1)} = \xi^{(k)} + \Delta\xi^{(k)}; \quad \eta_{ij}^{(k+1)} = \eta_{ij}^{(k)} + \Delta\eta_{ij}^{(k)} \tag{17}$$

$$\tilde{\mathbf{r}}_A^{(k+1)} = \Delta\tilde{\mathbf{u}}_A^{(k)} + \Delta\lambda^{k+1}\tilde{\mathbf{m}}_A^{(k+1)} \tag{18}$$

Set the iterative step $(k + 1) \rightarrow (k)$, go to convergence check (see Section 3.2).

3.5. Converged state variables

The converged internal state variables are

$${}^{n+1}F_{ij}^{p} = \left(\delta_{ik} + \Delta\lambda^{(k+1)} \; {}^{n+1}\bar{M}_{ik}^{(k+1)}\right) {}^{n}F_{kj}^{p} \tag{19}$$

$${}^{n+1}\xi = {}^{n+1}\,\xi^{(k+1)}; \; {}^{n+1}\eta_{ij} = {}^{n+1}\,\xi_{ij}^{(k+1)}; \; {}^{n+1}q = {}^{n+1}\,q^{(k+1)}; \quad {}^{n+1}a_{ij} = {}^{n+1}\,q_{ij}^{(k+1)} \tag{20}$$

The updated known variables at a this time step are then $\{{}^{n+1}F_{ij}^{p}, {}^{n+1}\xi, {}^{n+1}q, {}^{n+1}\eta_{ij}, {}^{n+1}a_{ij}\}$.

It should be mentioned that the explicit formulation on the corresponding algorithmic tangent stiffness, either for the total Lagrangian approach or for the updated Lagrangian approach, can be obtained for this algorithm by simple mathematical derivation. It is omitted in this paper due to space limitations.

4. Numerical examples

The first example is concerned with uniaxial loading. The von Mises yield and potential surfaces and linear isotropic hardening law define the model. The tension ratio is calculated up to 1.4 to show large deformations but using only 40 time steps to show the robustness of the algorithm. Typical energy norm values versus number of iteration are given in Fig. 1. The convergence rates are very fast; in fact, there are up to only four iteration steps for this problem. This is to be expected, as von Mises yield and potential functions are quadratic in stress, so the Newton method should have an infinitely large trust region [28]. The stress–strain result for the monotonic and cyclic responses are given in Fig. 2.

Another example presented is simple shear loading. The hardening was changed from isotropic to linear kinematic. This time, the induced anisotropy of kinematic hardening was evaluated. The anisotropic hardening of von Mises yield and potential surface will be reflected in induced (functional) anisotropy of those two functions in stress space. The simple shear ratio is calculated up to 0.4. The results of monotonic and cyclic response response are given in Fig. 3.

5. Conclusions

In this paper, we have presented a return mapping algorithm for general hyperelastoplastic large deformations in the framework of multiplicative decomposition. The developments are not restricted to isotropy and therefore can handle with ease anisotropic materials, including anisotropic hyperelasticity, anisotropic yield and plastic flow functions, and anisotropic hardening/softening laws (kinematic and distortional). The implementation of the algorithmic tangent stiffness provides very fast convergence rates within Newton's trust region, as shown in the examples.

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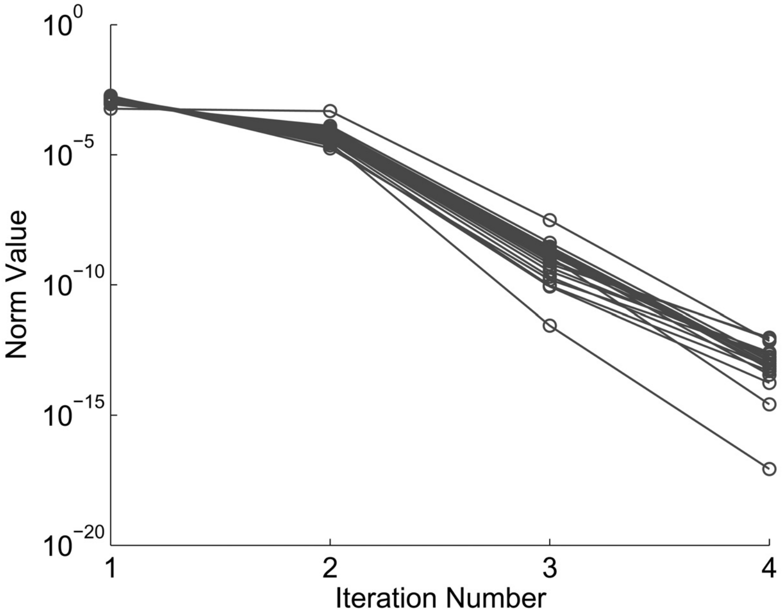

Fig. 1. Typical convergence rates in plasticity: norm values versus iteration numbers.

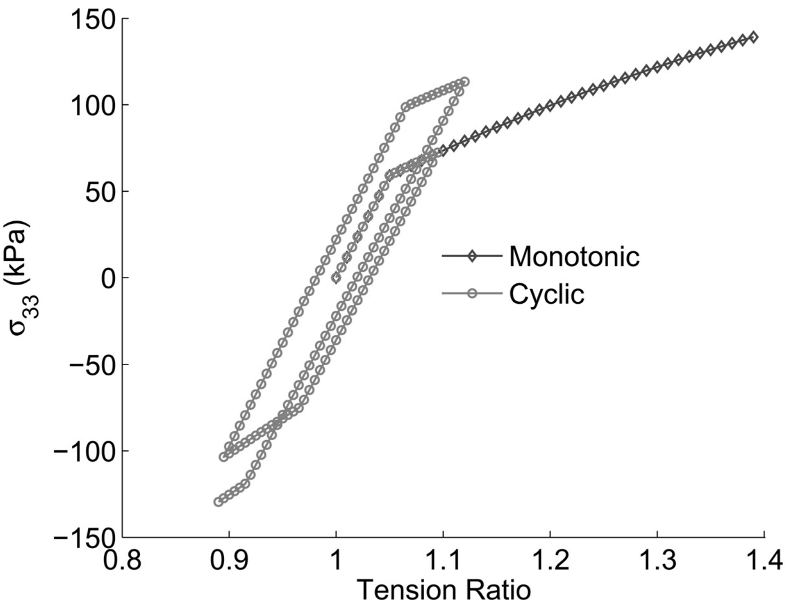

Fig. 2. Uniaxial loading, compressible neo-Hookean hyperelastic, shear modulus 442.5 kPa, bulk modulus 1976.67 kPa, linear isotropic hardening law with hardening modulus 500 kPa, initial yield strength 60 kPa: uniaxial Cauchy stresses versus tension ratio, monotonic and cyclic response.

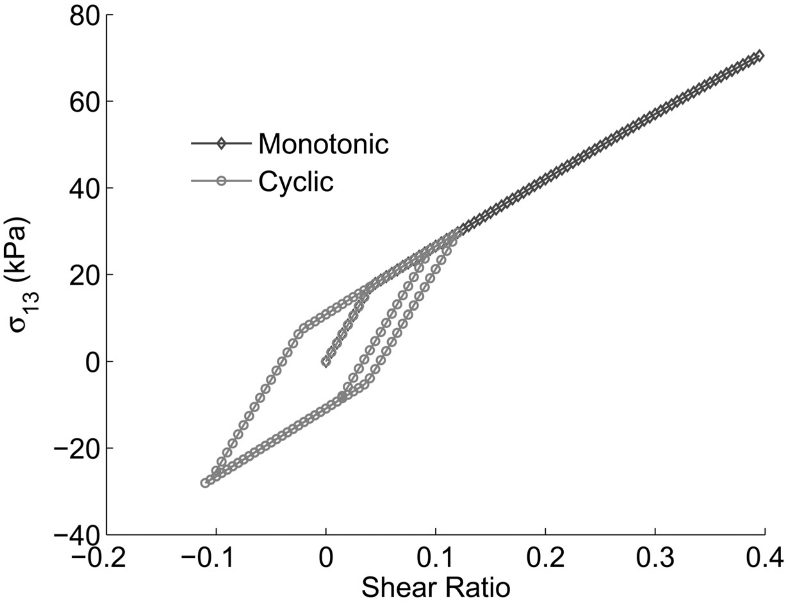

Fig. 3. Simple shear compressible neo-Hookean hyperelastic, shear modulus 442.5 kPa, bulk modulus 1976.67 kPa, linear kinematic hardening law with hardening modulus 500 kPa, initial yield strength 30 kPa: shear Cauchy stresses versus shear ratio, monotonic and cyclic response.

Note

[1] Indicial notation and summation convention for vectors and tensors as well as Cartesian coordinates are used in this paper.

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