High-accuracy numerical methods for incompletely parabolic problems in fluid dynamics I: formulation

Maria Morandi Cecchi[a,*], Maria A. Pirozzi[b]

[a] Dipartimento di Matematica Pura e Applicata, University of Padova, Italy
[b] Istituto di Matematica, Fisica e Applicazioni, University of Napoli 'Parthenope', Italy

Abstract

We continue to investigate a family of fully discrete finite difference implicit methods already proposed for the numerical solution of multidimensional hyperbolic and convection–diffusion equations. In this paper the extension of the schemes to the resolution of two-dimensional incompletely parabolic problems is considered. The basic idea is to discretize and then to split the resulting system of algebraic equations into a collection of tridiagonal subsystems associated with each gridline. The truncation error analysis leads to conditions on the order of accuracy. The classical Von Neumann method is applied to assess the stability of the schemes which is guaranteed with no restriction on the time step.

Keywords: Mixed hyperbolic–parabolic systems; Finite-difference; High-order schemes; Splitting methods

1. Introduction

The construction, analysis and implementation of approximate solution of nonlinear conservation laws and related equations were the major focus for an enormous amount of activity in recent decades.

For the numerical solution of multidimensional hyperbolic and convection–diffusion problems a family of fully discrete finite difference implicit high-order schemes with two time level and three-point support has been recently proposed and studied in Pirozzi [1] and Morandi Cecchi et al. [2]. The schemes are unconditionally stable, solvable and satisfying the condition for strict diagonal dominance. The maximum order of accuracy possible is third-order in space and second-order in time. A wide series of computational experiments illustrate and validate the behavior of the schemes. The numerical results show that the family has good performance in stability and accuracy.

The aim of this paper is to investigate the application of the family to the resolution of multidimensional incompletely parabolic systems of partial differential equations

$$\frac{\partial w}{\partial t} = \sum_{j,k=1}^{d} B^{(jk)} \frac{\partial^2 w}{\partial x_j \partial x_k} + \sum_{j=1}^{d} A^{(j)} \frac{\partial w}{dx_j} + Cw + F, \quad d = 2 \text{ or } 3 \tag{1}$$

where $w = (w_1, \ldots, w_r)^T$ is a vector function of $x_1, \ldots, x_d$, t and where $A^{(j)}$, $B^{(jk)}$ and C are square matrices of order r. Following Gustafsson et al. [3] the matrices $B^{(jk)}$ are rank deficit with some rank $s < r$ and should be jointly transformable to the form

$$B^{(jk)} = \begin{bmatrix} B_1^{(jk)} & 0 \\ 0 & 0 \end{bmatrix}$$

For simplicity only, we assume that the matrices $B^{(jk)}$ already have this form and that $w = (w^I, w^{II})^T$ and

$$A^{(j)} = \begin{bmatrix} A_{11}^{(j)} & A_{12}^{(j)} \\ A_{12}^{(j)} & A_{22}^{(j)} \end{bmatrix}$$

are partitioned in the same way as $B^{(jk)}$. We now require that the system

$$\frac{\partial w^I}{\partial t} = \sum_{j,k} B^{(jk)} \frac{\partial^2 w^I}{\partial w_j \partial x_k}$$

be parabolic and that the system

* Corresponding author. Tel.: +39 049 827 5904; Fax: +39 049 827 5843; E-mail: mcecchi@math.unipd.it

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$$\frac{\partial w^{II}}{\partial t} = \sum_j A_{22}^{(j)} \frac{\partial w^{II}}{\partial x_j}$$

be strictly hyperbolic. It is to be noted that the compressible Navier–Stokes equations and the viscous shallow-water equations in two space dimensions are of the incompletely parabolic type.

In practical applications one studies flows in finite domains and Eq. (1) has to be supplemented by appropriate initial and boundary conditions which lead to a mathematically well-posed problem, as examined in Gustafsson et al. [3].

Multidimensional implicit schemes are often more effective than explicit schemes, primarily due to their inherently more stable behavior, particularly if the accuracy of the spatial solution is more critical than that of the temporal solution. However, implicit methods have the problem of achieving an efficient solution algorithm. Therefore, special procedures are necessary which are often built around some means of splitting the equations on a convenient coordinate basis.

As related in Morandi Cecchi et al. [2], the basic idea behind the splitting methods is to separate the operators into one-dimensional components and split the scheme into two or three steps, each one involving only the implicit operations originating from a single coordinate. In the case at hand, the overall strategy is to discretize and then to manipulate the resulting algebraic equations to generate a collection of tridiagonal subsystems associated with each gridline. Theoretical accuracy estimates are given and a Von Neumann analysis indicates that the schemes are unconditionally stable.

2. Construction of the schemes

To begin with, let us consider the following two-dimensional symmetric linear constant coefficient version of the system (1)

$$\frac{\partial w}{\partial t} = B \sum_{j=1}^{d} \frac{\partial^2 w}{\partial^2 x_j} + \sum_{j=1}^{d} A^{(j)} \frac{\partial w}{\partial x_j} \tag{2}$$

where

$$B = \mathrm{diag}[v,v,0], \quad A^{(1)} = -\begin{bmatrix} 1 & 0 & c \\ 0 & 1 & 0 \\ c & 0 & 1 \end{bmatrix},$$

$$A^{(2)} = -\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & c \\ 0 & c & 1 \end{bmatrix}, \; v > 0 \text{ and } c > 0 \tag{3}$$

We introduce a difference grid which is uniform in $\mathbf{x} = (x, y)$ and t

$$x_m = m\Delta x, \quad y_s = s\Delta y, \quad t_n = n\Delta t, \quad \Delta x = \frac{L}{M+1},$$
$$\Delta y = \frac{L}{S+1}, \quad \Delta t = \frac{T}{N+1}$$

For all functions defined at the grid nodes let

$$\Phi(n,m,s) = \phi(x_m, y_s, t_n), \quad m = 0,1,\ldots,M+1;$$
$$s = 0,1,\ldots,S+1; \quad n = 0,1,\ldots,N+1$$

Looking for, at least, second order schemes in space and time, a natural extension of the family examined in [1] is

$$M_1 \Delta \mathbf{W}^* = \sum_{j=1}^{d} C_j \mathbf{W}_j(n,m,s) \tag{4}$$

$$M_2 \Delta \mathbf{W} = M_2[\mathbf{W}(n+1,m,s) - \mathbf{W}(n,m,s)] = \Delta \mathbf{W}^* \tag{5}$$

with

$$M_j = Q_j - \frac{1}{2}[\Gamma_j A^{(j)} + \sigma_j \delta_j^2 B] = Q_j - \frac{R_j}{2},$$
$$C_j = Q_{1..} Q_{j-1} R_j Q_{j+1..} Q_d \tag{6}$$

$$Q_j = I + a_j \frac{\delta_j^2}{2} - \frac{\delta_j^0}{4} K_j, \quad \Gamma_j = \frac{\tau_j}{2}[\delta_j^0 - \delta_j^2 K_j], \quad 1 \le j \le d \tag{7}$$

Here

$$\delta_j^- = 1 - E_j^{-1}, \quad \delta_j^+ = E_j - 1, \quad \delta_j^0 = \delta_j^+ + \delta_j^-,$$
$$\delta_j^2 = \delta_j^- \delta_j^+$$

E_j stands for the forward shift operator acting on the variable indicated as subscript

$$\tau_j = \frac{\Delta t}{\Delta x_j}, \quad \sigma_j = \frac{\tau_j}{\Delta x_j} \tag{8}$$

$$K_j = S_j D_j S_j^T, \quad D_j = \mathrm{diag}\,[\mathrm{sgn}(\lambda_{j,r})],$$
$$\Lambda_j = S_j^T A^{(j)} S_j = \mathrm{diag}\,[\lambda_{j,r}] \tag{9}$$

where Λ_j, S_j are the characteristic data of matrices $\mathrm{A}^{(j)}$ given in Eq. (3). Clearly, the operators Q spread the influence of the operators R in the normal direction and $\Delta\mathbf{W}^*$ can be thought of as the correction to the solution at time-level n.

The combination of Eqs. (4) and (5) results in a method that involves a nine-point cluster of space points centred about the point (m,s) and advances the solution from $n\Delta t$ to $(\mathrm{n}+1)\Delta t$

$$M^* \Delta \mathbf{W} = \sum_{j=1}^{d} C_j \mathbf{W}_j(n,m,s), \quad M^* = \prod_{j=1}^{d} M_j \tag{10}$$

To be noted that algebraic operators appropriate to

both directions appear on the left-hand side of Eq. (10). Therefore, it can be implemented very efficiently as a two-stage algorithm at each time-step. Because each factor is treated separately, as related in Pirozzi [4] an important property can be readily observed, namely that the systems are always diagonal dominant provided that the parameters a_j in Eq. (7) satisfy the inequality

$$a_j < \frac{1}{2} - \sigma_j \nu \tag{11}$$

with σ_j given by Eq. (8). The accuracy of the combined method (10) may be determined directly. Performing a Taylor expansion in the same way as in the one-dimensional case, we obtain that the scheme has a local truncation error of order $O(\Delta t^2 + \Delta x^2 + \Delta y^2 + q_1 + q_2)$ for $q_j = (\nu \Delta x)_j$. Moreover, for

$$a_j = \frac{1}{3} \tag{12}$$

the scheme has a local truncation error of the form $O(\Delta t^2 + \Delta x^3 + \Delta y^3 + q_1 + q_2)$. Clearly, the meaning of such an approximation consists in the following. Terms of order $O(\Delta x + \Delta y)$ appear as a result of introducing the operator δ^2 for the viscous terms. However, a more exact representation of the solution is possible as compared to lower-order schemes.

The classical Von Neumann method is applied to assess the conditions for the stability of the schemes and also for the solvability of the implicit operators. When Fourier modes are substituted in Eq. (10), we have

$$H[\mathbf{W}^a_{n+1} - \mathbf{W}^a_n] = \Theta \mathbf{W}^a_n$$

where

$$H = \prod_{j=1}^{d} \left(I + \Psi_j - \frac{\tau_j}{2} A^{(j)} T_j + B\sigma_j (1 - \cos \eta_j) \right)$$

$$\Theta = \tau_1 A^{(1)} T_1 (I + \Psi_2) + \tau_2 (I + \Psi_1) T_2 A^{(2)} - 2\sigma_1 B (I + \Psi_2)(1 - \cos \eta_1) - 2\sigma_2 (I + \Psi_1) B (1 - \cos \eta_2)$$

$$\Psi_j = (1 - \cos \eta_j) a_j I - i \sin \eta_j \frac{K_j}{2},$$

$$T_j = (1 - \cos \eta_j) K_j + i \sin \eta_j I$$

τ_j, σ_j, K_j are given in Eqs. (8)–(9), $\eta_j = k_j \Delta x_j$ are the dimensionless wave numbers of a Fourier component $\exp[i(k_1 x + k_2 y)]$ and $\mathbf{W}^a_n$ is the amplitude function for the approximate solution $\mathbf{W}(n,m,s)$. A first condition to be imposed on the implicit operator is that the matrix H should not vanish. This ensures that the scheme will always be solvable. This will be the case when the inequality (11) holds. Thus, the amplification matrix $\mathbf{G}$ of the combined method is given by

$$G = G(k_1, k_2) = I + H^{-1} \Theta$$

As the structure of the matrix $\mathbf{G}$ is quite complicated, analytical Von Neumann stability conditions cannot be derived and so numerical investigations are needed. We have to compute the maximum of the spectral radius of G over a discrete set of points k_1, k_2 in $[0,2\pi] \times [0,2\pi]$. By doing this for different values of the mesh ratio $\Delta t / \Delta x = \Delta t / \Delta y$ and, by observing when this quantity exceeds 1, it is possible to select the parameters in Eq. (7) in order to guarantee unconditional stability.

The scheme defined by Eqs. (4)–(5) may be extended in a straightforward way to nonlinear problems by considering the operators

$$R_j = \Gamma_j A^{(j)} + \sigma_j \delta_j^- (E_j^{1/2} B) \delta_j^+,$$

$$\Gamma_j = \frac{\tau_j}{2} \left(\delta_j^0 - \delta_j^- (E_j^{1/2} K_j) \delta_j^+ \right)$$

Clearly, as in the hyperbolic case, to mantain the highest-order spatial accuracy for the scheme satisfying Eq. (12), we have to supplement the right-hand side of Eq. (4) with additional terms that take into account the non-commutative nature of the operators involved.

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