Discontinuous Galerkin approximation of eigenvalue problems

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Abstract

In this paper we analyse the problem of computing eigenvalues and eigenfunotions of the Laplace operator and of the grad-div operator by discontinuous Galerkin (DG) methods. Conditions under which DG methods provide spectrally correct approximations are given in both cases.

Keywords: Discontinuous Galerkin methods; Eigenvalue problems

1. Introduction

In this paper we address the question of whether discontinuous Galerkin (DG) methods can be used for spectral computations. We consider the cases of the Laplace operator (with compact inverse), and of the grad-div operator (with non-compact inverse), which describe the vibration frequencies of a prestressed membrane and of a fluid in a cavity, respectively. We show that, in both cases, the answer to this question is affirmative for a wide class of (stable) DG methods. Our spectral approximation analysis takes inspiration from previous works on conforming discretizations as, e.g., [1,2,3,4]. The results presented in this paper are developed in [5] and [6].

2. The Laplace eigenproblem

Consider the eigenvalue problem

$$-\Delta u = \lambda u \ \text{ in } \Omega, \qquad u = 0 \ \text{ on } \partial\Omega \tag{1}$$

where Ω is a bounded Lipschitz polygonal domain in $\mathbb{R}^2$.

Set $V = H^1_0(\Omega)$ and $\|u\|_V = \|\nabla u\| L^2(\Omega)^2$. From here on, we denote by $(\cdot,\cdot)$ the standard inner product in $L^2(\Omega)^n$, $n = 1, 2$. The variational form of problem (1) consists in finding $(0 \neq u, \lambda) \in V \times \mathbb{R}$ such that

$$a(u, v) := (\nabla u, \nabla v) = \lambda(u, v) \quad \forall v \in V$$

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Consider any DG method (see, e.g., [7]) defined on regular and shape-regular meshes T_h, with DG bilinear from a_h (.,.) defined on the DG space:

$$V_h := \{v \in L^2(\Omega) : v|_K \in \mathcal{P}^\ell(K) \quad \forall K \in \mathcal{T}_h\}$$

where $\mathcal{P}^\ell(K)$ is the space of polynomials of degree at most $\ell \geq 1$ on K. The discrete problem is then defined as: find $(0 \neq u_h, \lambda_h) \in V_h \times \mathbb{R}$ such that

$$a_h(u_h, v) = \lambda_h(u_h, v) \quad \forall v \in V_h$$

Introduce $V(h) := V + V_h$, endowed with the following norm:

$$\|v\|^2_{V(h)} = \sum_{K \in \mathcal{T}_h} \|\nabla v\|^2_{L^2(K)^2} + \sum_{E \in \mathcal{E}_h} \left\| h^{-\frac{1}{2}} [\![v]\!] \right\|^2_{L^2(E)}$$

where ε_h is the set of all edges of $\mathcal{T}_h$ and $[\![v]\!]$ is the jump of v across E. The following Poincaré-type inequality holds true [7]: for all $f \in V(h)$, $\|f\|_{L^2(\Omega)} \leq C\|f\|_{V(h)}$, where the constant C depends only on Ω but not on the mesh.

The essential condition a DG method has to satisfy to ensure spectral correct ness is the following property:

Property 1 (quasi-optimality) Let $f \in L^2(\Omega)$ and $u_s \in V$ be the solution of $-\Delta u_s = f$ in Ω. The DG method for the source problem: find $u_h \in V_h$ such that $a_h(u_h, v_h) = (f, v_h)$ for all $v_h \in V_h$ defines a unique discrete solution and the following quasi-optimal error estimate holds true:

$$\|u_s - u_h\|_{V(h)} \leq Ch^t \|u_s\|_{H^{t+1}(\Omega)} \qquad t = \min\{\ell, \sigma\}$$

where σ is the elliptic regularity exponent of the source problem (i.e. for any given $f \in L^2(\Omega)$, the solution $u_s \in V$

of $-\Delta u_s = f$ in Ω satisfies $u_s \in H^{\sigma+1}(\Omega)$ and $\|u_s\|_{H^{\sigma+1}(\Omega)} \leq C\|f\|_{L^2\Omega)}$).

From now on we assume that Property 1 holds true.

We define the following continuous and discrete solution operators:

$$T : L^2(\Omega) \to V \qquad a(Tf,v) = (f,v) \qquad \forall v \in V$$
$$T_h^{DG} : L^2(\Omega) \to V_h \qquad a_h(T_h^{DG}f,v) = (f,v) \qquad \forall v \in V_h$$

and denote by $\sigma(T)$ and $\sigma(T_h^{DG})$ the spectrum of T and T_h^{DG}, respectively.

Theorem 1 (non-pollution of the spectrum) Let $A \subset \mathbb{C}$ be an open set containing $\sigma(T)$. Then, for h small enough, $\sigma(T_h^{DG}) \subset A$.

Introduce the following notion of 'distance' (see [8]): for any Y and Z closed subspaces of $V(h)$,

$$\delta_h(Y,Z) := \sup_{\substack{y\in Y \\ \|y\|_{V(h)}=1}} \inf_{z\in Z} \|y - z\| v(h),$$

$$\hat{\delta}_h(Y,Z) := \max\{\delta_h(Y,Z), \delta_h(Z,Y)\}$$

Theorem 2 (Completness of the spectrum and convergence) Let μ be an eigenvalue of T with algebraic multiplicity n. Then, for h small enough, there exist exactly n eigenvalues $\{\mu_{1,h}, \ldots, \mu_{n,h}\}$ of T_h^{DG} (repeated with their multiplicities) which converge to μ and

$$\sup_{1\leq i\leq n} |\mu - \mu_{i,h}| \leq C(\delta_h(E(\mu),V_h))^2$$

where $E(\mu)$ is the eigenspace associated with μ. Moreover, let $E_h(\mu)$ be the sum of the eigenspaces associated with $\mu_{1,h}, \ldots, \mu_{n,h}$. Then:

$$\hat{\delta}_h(E(\mu),E_h(\mu)) \leq C\delta_h(E(\mu),V_h)$$

Note that $\delta_h(E(\mu), V_h)$ is the approximation error for eigenfunctions in V_h.

3. The grad-div eigenproblem

Consider the eigenvalue problem

$$\begin{aligned} -\nabla(\nabla\cdot\mathbf{u}) &= \omega^2\mathbf{u} && \text{in } \Omega \\ \text{rot } \mathbf{u} &= 0 && \text{in } \Omega \\ \mathbf{u}\cdot\mathbf{n} &= 0 && \text{on } \partial\Omega \end{aligned} \tag{2}$$

where Ω is a bounded Lipschitz polygonal domain in $\mathbb{R}^2$, and $\mathbf{n}$ is the outward normal unit vector to $\partial\Omega$. We recall that the 2D rotational operators rot and **rot** are defined by rot $\mathbf{v} = \partial_1 v_2 - \partial_2 v_1$ and **rot** $w = (\partial_2 w, -\partial_2 w, -\partial_1 w)$, respectively.

One of the standard ways to discretize the eigenproblem (2) consists in neglecting the constraint rot $\boldsymbol{u} = 0$ and add a zero frequency corresponding to the *infinite-dimensional* kernel of the grad-div operator. In variational formulation: find $(\mathbf{0} \neq \boldsymbol{u}, \lambda) \in \boldsymbol{V} \times \mathbb{R}$, with $\boldsymbol{V} := H_0(\text{div};\Omega)$, such that

$$a(\boldsymbol{u},\boldsymbol{v}) := (\nabla\cdot\boldsymbol{u}, \nabla\cdot\boldsymbol{v}) = \lambda(\boldsymbol{u},\boldsymbol{v}) \qquad \forall \boldsymbol{v} \in V \tag{3}$$

Following this approach, denoting by $a_h(\cdot,\cdot)$ the DG-bilinear form obtained by discretizing the grad-div operator by any DG method with DG method with DG space $\boldsymbol{V}_h := (V_h)^2$, the discrete problem to be solved reads: find $(\mathbf{0} \neq \boldsymbol{u}_h, \lambda_h) \in \boldsymbol{V}_h \times \mathbb{R}$ such that

$$a_h(\boldsymbol{u}_h,\boldsymbol{v}) = \lambda_h(\boldsymbol{u}_h,\boldsymbol{v}) \qquad \forall \boldsymbol{v} \in \boldsymbol{V}_h$$

Let $\boldsymbol{V}(h) = \boldsymbol{V} + \boldsymbol{V}_h$ be endowed with the semi-norm and norm:

$$|\boldsymbol{v}|^2_{V(h)} := \sum_{K\in\mathcal{T}_h} \|\nabla\cdot\boldsymbol{v}\|^2_{L^2(K)} + \sum_{E\in\mathcal{E}_h} \left\| h^{-\frac{1}{2}}[\![\boldsymbol{v}]\!]_N \right\|^2_{L^2(E)},$$

$$\|\boldsymbol{v}\|^2_{V(h)} = \|\boldsymbol{v}\|^2_{L^2(\Omega)^2} + |\boldsymbol{v}|^2_{V(h)},$$

where $[\![\mathrm{u}]\!]_N$ is the jump of the normal component of $\boldsymbol{v}$ across E.

The analysis essentially follows [2] and [4], plus treatment of non-conformity. It can be proved that a DG discretization of (3) is spectrally correct in the sense made precise in [2] (see also [4]), if the following properties are verified:

Property 2 (Coercivity in seminorm) For all $\boldsymbol{v} \in \boldsymbol{V}_h$, $a_h(\boldsymbol{v}, \boldsymbol{v}) \geq C|\boldsymbol{v}|^2_{V(h)}$, with the constant C independent of the mesh size.

Property 3 (Compatibility of the discrete kernel) Let $Q_h^c := \{q \in H_0^1(\Omega): q|_K \in \mathcal{P}^{\ell+1}(K) \;\; \forall K \in \mathcal{T}_h\}$, and denote by K_h the discrete kernel of $a_h(\cdot,\cdot)$ in $\boldsymbol{V}_h$. Then, **rot** $Q_h^c \subseteq K_h$.

Property 4 (Quasi-optimality) Let $\boldsymbol{f} \in L^2(\Omega)^2$ and $\boldsymbol{u}_s \in \boldsymbol{V}$ be the solution of $-\nabla(\nabla\cdot\boldsymbol{u}_s) + \boldsymbol{u}_s) = \boldsymbol{f}$ in Ω. The corresponding DG method: find $\boldsymbol{u}_h \in \boldsymbol{V}_h$ such that $a_h(\boldsymbol{u}_h, \boldsymbol{v}) + (\boldsymbol{u}_h, \boldsymbol{v}) = (\boldsymbol{f}, \boldsymbol{v})$ for all $\boldsymbol{v} \in \boldsymbol{V}_h$ defines a unique discrete solution. Moreover, whenever rot $\boldsymbol{f} = 0$, the following quasioptimal error estimate holds true:

$$\|\boldsymbol{u}_s - \boldsymbol{u}_h\|_{V(h)} \leq Ch^t\left(\|\boldsymbol{u}_s\|_{H^t(\Omega)^2} + \|\nabla\cdot\boldsymbol{u}_s\|_{H^t(\Omega)}\right)$$
$$t = \min\{\ell,\sigma\}$$

where σ is the regularity exponent of the source problem (i.e. for any given $\boldsymbol{f} \in L^2(\Omega)^2$, the solution $\boldsymbol{u}_s \in \boldsymbol{V}$ of $-\nabla(\nabla\cdot\boldsymbol{u}_s) + \boldsymbol{u}_s = \boldsymbol{f}$ in Ω satisfies $\boldsymbol{u}_s \in H^\sigma(\Omega)^2$, $\nabla\cdot\boldsymbol{u}_s \in H^\sigma(\Omega)$ and $\|\boldsymbol{u}_s\|_{H^\sigma(\Omega)^2} + \|\nabla\cdot\boldsymbol{u}_s\|_{H^\sigma(\Omega)} \leq C\|\boldsymbol{f}\|_{L^2(\Omega)^2}$).

4. Conclusions

We have addressed the problem of spectral correctness of DG methods as nonconforming approximations of problems (1) and (2). The theory presented in Section

2 can be extended to three dimensions and to meshes with hanging nodes, whereas possible extensions of the results announced in Section 3 are under investigation.

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