Superconvergence of the local discontinuous Galerkin method applied to diffusion problems

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Abstract

We present new superconvergence results for the local discontinuous Galerkin method applied to transient diffusion problems and examine the effect of numerical fluxes on superconvergence. We show that the gradient of the p-degree discontinuous finite element solution is superconvergent at the roots of the derivative of $(p + 1)$-degree Radau polynomial.

Keywords: Superconvergence; Finite elements; Discontinuous Galerkin; Diffusion problems

1. Introduction

Discontinuous Galerkin (DG) methods have gained much popularity in the past 15 years. For a detailed discussion of the history of DG method and a list of important citations on the DG method and its applications consult [1]. The success of DG methods is due to the following properties: (i) they do not require continuity across element boundaries; (ii) they are locally conservative; (iii) they are well suited to solve problems on locally refined meshes with hanging nodes; (iv) they have a very simple communication pattern between elements which makes them ideal for parallel computations; and (v) they exhibit strong superconvergence of solutions and fluxes for hyperbolic [2,3,4], elliptic [5] and convection-diffusion [6] problems.

The local discontinuous Galerkin (LDG) finite element method for solving convection–diffusion partial differential equations was introduced in [7]. Castillo [5] showed that on each element the p – degree LDG solution gradient is $O(\Delta x^{p+1})$ superconvergent at the shifted roots of the p – degree Legendre polynomial. Adjerid *et al.* [6] showed that the LDG method of Cockburn and Shu [7] for time dependent convection–diffusion problems exhibits $O(\Delta x^{p+2})$ superconvergence of the solution at the shifted Radau points on each element. For diffusion-dominated problems, they further showed that the derivative of the LDG solution is $O(\Delta x^{p+2})$ superconvergent at the roots of the derivative of Radau polynomial of degree $p + 1$.

In this manuscript we examine the superconvergence of LDG solutions for a family of numerical fluxes where the numerical flux considered in [6] is a special case. This manuscript is organized as follows: in Section 2 we present a model problem and recall the LDG formulation. In Section 3 we present numerical results for a one-dimensional linear diffusion problem. We conclude with a discussion of our findings in Section 4.

2. The local discontinuous Galerkin method

The local discontinuous Galerkin method for convection-diffusion problems was introduced by Cockburn in [7] and several a priori error estimates have been established for linear problems [7,8,9]. Here we consider the scalar convection-diffusion problem

$$u_t - du_{xx} = f, \quad a < x < b, t > 0, \quad d > 0 \tag{1a}$$

subject to the initial and boundary conditions

$$u(x,0) = u_0(x), a < x < b, \quad u(a,t) = u_a(t), \quad u(b,t) = u_b(t), t > 0 \tag{1b}$$

Following [7], we introduce the auxiliary variable $q = \sqrt{d}u_x$ to define the flux function

$$\mathbf{h} = (h_u, h_q)^T = (-\sqrt{d}q, -g(u)), \quad g(u) = \sqrt{d}u \tag{2a}$$

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and write Eq. (1) as:

$$u_t + (h_u)_x = f, \quad q + (h_q)_x = 0, \quad a < x < b, t > 0 \quad (2b)$$

In the remainder of this paper we shall use the notation $\mathbf{w} = (u, q)^t$.

Let us partition $[a, b]$ into N subintervals $I_j = [x_{j-1}, x_j]$, $j = 1, \ldots, N$, with $\Delta x_j = x_j - x_{j-1}$ and $\Delta x = (b - a)/N$. The LDG weak formulation is obtained by multiplying Eq. (2b) by a test function (v, r) and integrating by parts to obtain

$$(u_t,v)_j - (h_u,v_x)_j + h_u v\Big|_{x_{j-1}^+}^{x_j^-} = (f,v)_j$$

$$(q,r)_j - (h_q,r_x)_j + h_q r\Big|_{x_{j-1}^+}^{x_j^-} = 0, \forall\ v, r \in H^1 \quad (3)$$

where the left and right limits are defined as $z(x_i^-) = \lim_{x\to x_i, x<x_i} z(x)$ and $z(x_i^+) = \lim_{x\to x_i, x>x_i} z(x)$, respectively.

The element inner product is defined as

$$(u,v)_j = \int_{x_{j-1}}^{x_j} uv dx$$

We construct a finite dimensional space $\mathcal{V}_N^p$ of discontinuous piecewise polynomial functions such that

$$\mathcal{V}_N^p = \{V | V|_{I_j} \in \mathcal{P}_p\} \quad (4)$$

where $\mathcal{P}_p$ denotes the space of polynomials of degree p.

The discrete LDG formulation consists of finding U and $Q \in \mathcal{V}_N^p$ such that

$$(U_t,V)_j - (h_U,V_x)_j + \hat{h}_U V\Big|_{x_{j-1}^+}^{x_j^-} = (f,V)_j \quad (5a)$$

$$(Q,R)_j - (h_Q,R_x)_j + \hat{h}_Q R\Big|_{x_{j-1}^+}^{x_j^-} = 0, \forall\ V, R \in \mathcal{V}_N^p \quad (5b)$$

If $P_p(\xi)$ is the Legendre polynomial of degree p, we shall refer to R_{p+1}^+ and R_{p+1}^- as the right and left $p + 1$-degree Radau polynomials, respectively, which are defined as

$$R_{p+1}^{\pm}(\xi) = P_{p+1}(\xi) \mp P_p(\xi), \ -1 \le \xi \le 1 \quad (6)$$

The weak problem (5) is subject to the initial condition $U(x, 0) \in \mathcal{V}_N^p$ obtained by interpolating u_0 on each interval at the shifted roots of R_{p+1}^+.

Since the trial function is discontinuous, the fluxes at the end points in (a, b) are replaced by the following numerical fluxes

$$\hat{h}(\mathbf{W}^-,\mathbf{W}^+) = (-\sqrt{d}\bar{Q}, -\sqrt{d}\bar{U})^t - \beta\frac{\sqrt{d}}{2}([Q], -[U])^t, \quad -1 \le \beta \le 1 \quad (7a)$$

where $\mathbf{W} = (U, Q)^t$, $[u] = u^+ - u^-$ and $\bar{u} = (u^+ + u^-)/2$.

The numerical flux at the boundary points is well defined by setting

$$(u,q)(a^-,t) = (u_a(t),q(a^+,t)), \ (u,q)(b^+,t) = (u_b(t), q(b^-,t)) \quad (7b)$$

and write the flux as

$$\hat{h}_U(b) = -\sqrt{d}Q(b^-) + \max\{1,p_N\}d/\Delta x_N\}(u(b,t) - U(b^-)), \text{ for } \beta = 1 \quad (7c)$$

$$\hat{h}_U(a) = -\sqrt{d}Q(a^+) + \max\{1,p_1\}d/\Delta x_1\}(U(a^+) - u(a,t)), \text{ for } \beta = -1 \quad (7d)$$

The function h_Q at the boundary points is obtained from (7a) and (7b).

Adjerid et al. [6] studied the case $\beta = 1$ and here we present results for the case $\beta = -1$.

3. A computational example

Let us consider the problem (1) with $d = 1$ on $(-1, 1)$ and select the boundary and initial conditions such that the exact solution is $u(x,t) = e^{-\pi^2 t}\sin(\pi x)$. We solve the problem on a 16-element uniform mesh using $p = 1$ to 4 and $0 \le t \le 0.5$ with $\beta = -1$. The errors shown in Figs 1 and 2 suggest that on (x_{i-1},x_i) we have

$$u(x,t) - U(x,t) = a_i(t) + b_i(t)R_{p+1}^-(x),$$
$$q(x,t) - Q(x,t) = c_i(t) + d_i(t)R_{p+1}^+(x) \quad (8)$$

Thus, the solution gradient is superconvergent at the shifted roots of $R_{p+1}^{-\prime}(x)$ while the derivative Q_x of the auxiliary variable is superconvergent at the roots of $R_{p+1}^{+\prime}(x)$. For $-1 < \beta < 1$ we did not observe any pointwise superconvergence.

4. Conclusion

Our computational results show that the derivative of the LDG solution is superconvergent at the shifted roots of $R_{p+1}^{\pm\prime}(x)$ for $\beta = \pm 1$, respectively. The results for $\beta = \pm 1$ are useful in computing efficient *a posteriori* error estimates that may help improve the the accuracy of the solution and/or steer the adaptive refinement process. We did not observe any pointwise superconvergence for $-1 < \beta < 1$. Currently, we are investigating superconvergence for nonlinear and multi-dimensional problems using rectangular and triangular meshes.

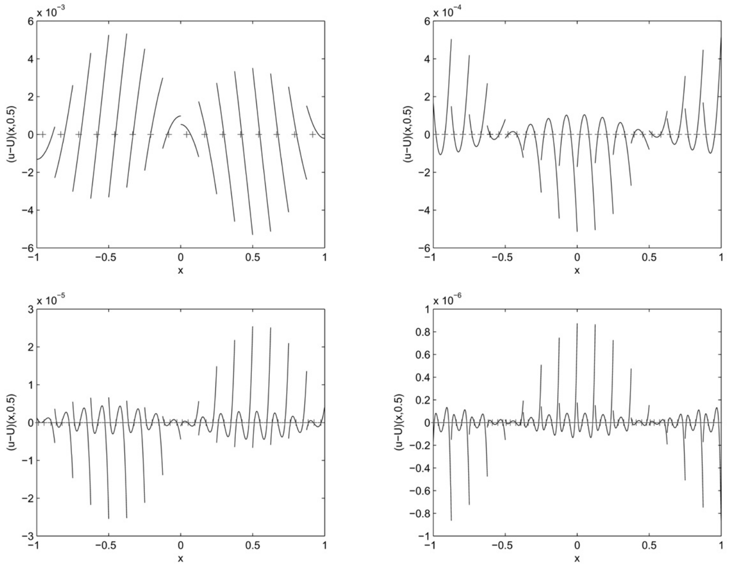

Fig. 1. True errors $(u - U)_x(x, 0.5)$ on a 16-element uniform mesh for $p = 1$ to 4 (upper left to lower right). The shifted roots of $R^{-\prime}_{p+1}(x)$ are marked by $+$.

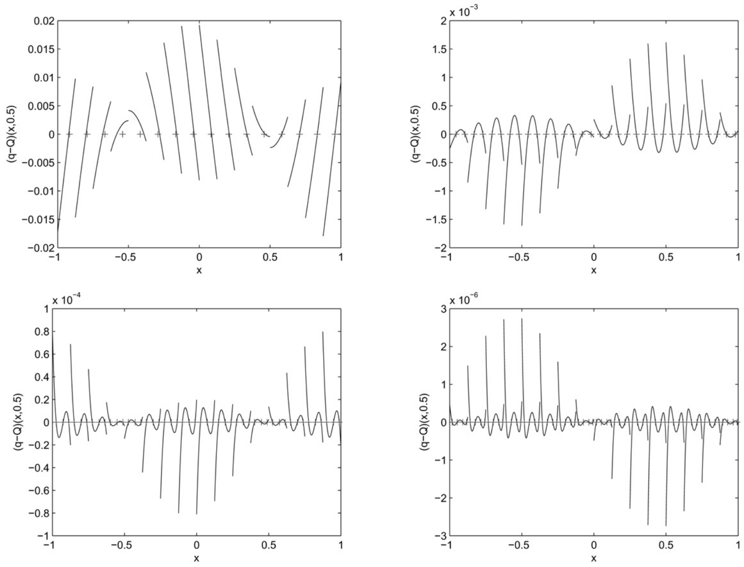

Fig. 2. True errors $(q - Q)_x(x, 0.5)$ on a 16-element uniform mesh for $p = 1$ to 4 (upper left to lower right). The shifted roots of $R^{+\prime}_{p+1}(x)$ are marked by $+$.

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