Efficient modeling of dispersive wave propagation in unbounded domains

M.A. Zahid, M.N. Guddati*

Department of Civil Engineering, North Carolina State University, Raleigh, NC 27695, USA

Abstract

Continued fraction absorbing boundary conditions (CFABC) are artificial boundary conditions that can be used for modeling wave absorption into an unbounded exterior. In this paper, CFABC are extended to modeling wave propagation in dispersive media. While the original CFABC are effective only for propagating waves, the proposed extension is effective also for evanescent waves that could be predominant in dispersive media. The main idea is to augment the original CFABC with a special padding region that would act as an effective boundary condition for evanescent waves. Presented here is the basic idea behind the extension, summary of various issues that were resolved during the development, and a numerical example illustrating the effectiveness of the proposed approach.

Keywords: Absorbing boundary conditions; Dispersive waves; Finite element methods

1. Introduction

Several problems of wave propagation involve domains of infinite extent where it is required to truncate the domain and use appropriate absorbing boundary conditions (ABCs) that simulate the absorption of waves into the exterior. The focus of the paper is to develop effective ABCs for dispersive media. Wave propagation in dispersive media is different (from non-dispersive media) in that, even in the absence of heterogeneities and physical boundaries, there exist evanescent waves in addition to propagating waves. In this paper, we extend continued fraction absorbing boundary conditions (CFABCs, [1]), effective boundary conditions for propagating waves, to evanescent waves, and thus to dispersive media.

CFABCs are based on a link between finite element discretization and rational approximations, and unify the two disparate classes of boundary conditions, namely the material absorbers (PML, [2]) and rational approximation based absorbers (see, e.g. [3]). CFABCs have the flexibility of material absorbers, while retaining the effectiveness of rational absorbers. Due to their flexibility, CFABCs can be extended to dispersive media, but such an extension is not straightforward. In addition to the modification required to handle evanescent waves, special time-stepping schemes are designed to solve a peculiar fourth-order differential equation. With these modifications, the resulting CFABCs become highly effective for dispersive media. In what follows, we present the basic idea of CFABC, extension to dispersive media, issues involved in numerical implementation and an example to illustrate the performance.

2. Continued fraction absorbing boundary conditions (CFABC)

CFABCs are derived using a four-step procedure. The first step involves the discretization of the exterior using finite elements, which results in discretization error. In the second step, which is the key to the effectiveness of CFABC, a special integration scheme is utilized to eliminate the error in the impedance of the exterior. In the third step, to make the problem computationally tractable, the number of finite elements are truncated in the direction normal to the artificial boundary. The approximation resulting from the truncation can be quantified in terms of the reflection coefficient at the artificial boundary [1]:

* Corresponding author. Tel.: +1 (919) 515 7699; Fax: +1 (919) 515 7908; E-mail: mnguddat@eos.ncsu.edu

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$$R = \prod_{j=1}^{n} \left(\frac{k - 2i/L_j}{k + 2i/L_j} \right)^2 \tag{1}$$

where n is the number of absorbing layers, k is the wavenumber in the direction perpendicular to the boundary and L_j are the lengths of finite elements (in the direction normal to the boundary). It is clear from Eq. (1) that, in order to reduce the reflection coefficient for propagating waves (with real wavenumber), the element lengths must be imaginary, which constitutes the final step of CFABC. This idea is similar to the complex coordinate stretching of PML [4]. The reader is referred to [1] for further details.

3. CFABC for dispersive wave

A model problem for dispersive wave propagation can be represented by the Klein-Gordon equation:

$$-c^2(u_{xx} + u_{zz}) + u_{zz} + f^2 u = 0 \tag{2}$$

where c and f^2 are the wave velocity and dispersion parameter respectively. Since the above equation is amenable to finite element discretization similar to that of the non-dispersive wave equation, the ideas presented in the previous section are immediately applicable and the CFABC can be readily extended. However, it is important to note that, in dispersive media, there exist evanescent waves (with imaginary or complex wavenumber), in addition to propagating waves (with real wavenumber). Evanescent waves are often characterized using the decay parameter, $\sigma = ik$, and the propagating waves are characterized using phase velocities, $c^p = \omega/k$. The reflection coefficient in Eq. (1) can be rewritten in terms of c^p and σ as:

$$R = \prod \left(\frac{c^p - \omega L_j/2i}{c^p - \omega L_j/2i} \right)^2 \prod \left(\frac{\sigma - 2/L_j}{\sigma + 2/L_j} \right)^2 \tag{3}$$

In order to reduce R for propagating waves, we choose imaginary element lengths $L_j = 2ic_j / \omega$, where c_j are the parameters of the boundary condition and are called the reference phase velocities. To reduce R for evanescent waves, the lengths are chosen as real, $L_j = 2 / \sigma_j$, where σ_j are called the reference decay coefficients. Such usage of real lengths is essentially a special discretization of a region around the computational boundary. This region is referred to as the padding region and the resulting boundary condition is called the padded CFABC. The reflection coefficient can be written in terms of c_j and σ_j as:

$$R = \prod_{j=1}^{n} \left(\frac{c^p - c_j}{c^p + c_j} \right)^2 \prod_{j=1}^{m} \left(\frac{\sigma - \sigma_j}{\sigma + \sigma_j} \right)^2 \tag{4}$$

where n is the number of absorbing layers, and m is the number of padding layers.

The parameters c_j and σ_j govern the performance of the boundary condition and are problem dependent. Based on the analysis presented in [5], the following choices of parameters tend to give good results:

$$c^j = \frac{c}{\cos(\pi j/2n)} \tag{5}$$

$$\ln(\sigma^j) = (j/m + 1)\ln(\sigma_{\max}/\sigma_{\min}) + \ln(\sigma_{\min}) \tag{6}$$

The minimum value of the decay coefficient ($\sigma_{\min}$) is selected from the dispersion relation and the maximum value of the decay coefficient ($\sigma_{\max}$) is selected based on position of the load [5].

4. Finite element discretization and time stepping

Since the CFABC is based on finite element discretization, no special effort is needed to obtain the finite element mesh (a typical discretization of CFABC is shown in Fig. 1). However, there are two important differences. Special integration is needed for both padding and absorbing regions, and frequency-dependent imaginary element length is used for the absorbing elements. Due to the frequency dependence, the spatially discrete equations become a fourth-order evolution equation in the time domain:

$$\mathbf{MA} + \mathbf{CV} + \mathbf{KU} + \mathbf{RW} + \mathbf{SX} = \mathbf{F} \tag{7}$$

In the above, $\mathbf{U}$ is the displacement vector, $\mathbf{V} = \partial\mathbf{U}/\partial t$, $\mathbf{A} = \partial^2\mathbf{U}/\partial t^2$, $\mathbf{W} = \int\mathbf{U}dt$ and $\mathbf{X} = \int\mathbf{W}dt$. The above equation is significantly more complicated than the regular second-order equation encountered in standard

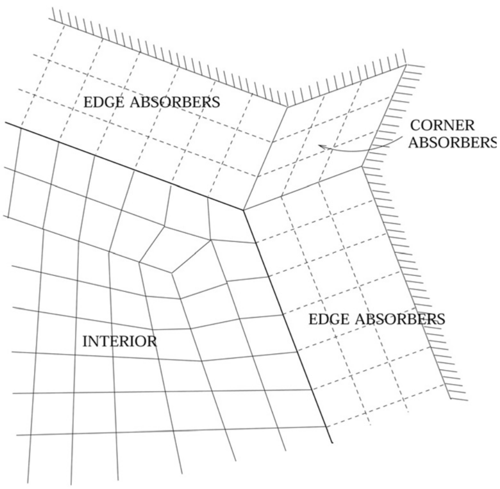

Fig. 1. Finite element discretization of CFABC.

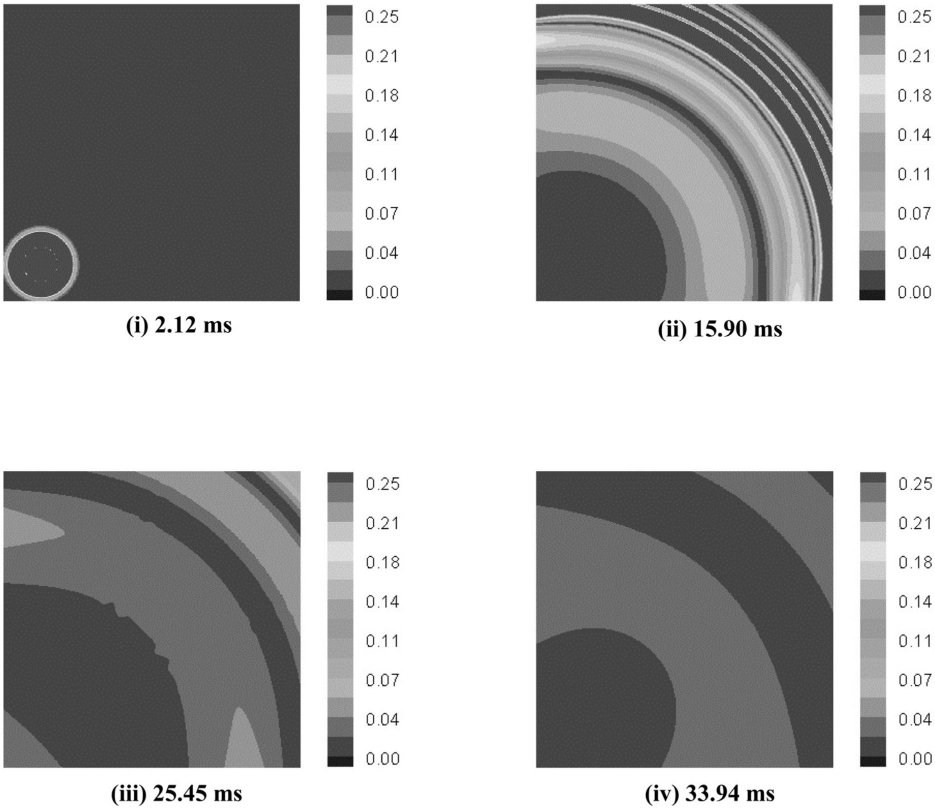

Fig. 2. CFABC solution with three absorbing layers.

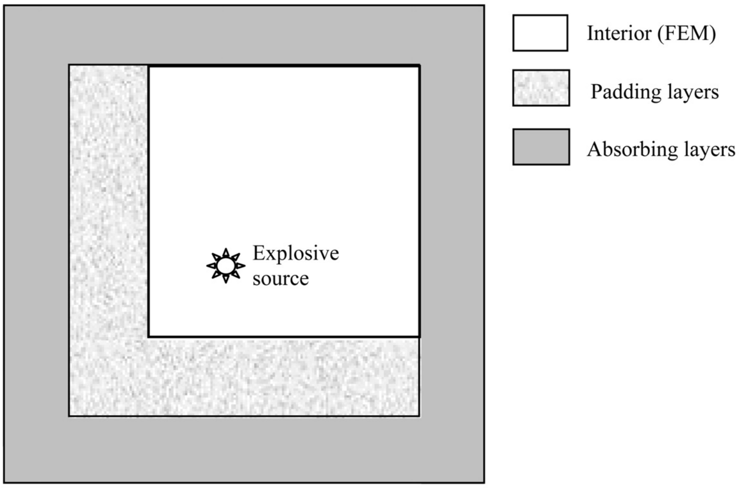

Fig. 3. Full space problem (30m × 30m) with padded CFABC.

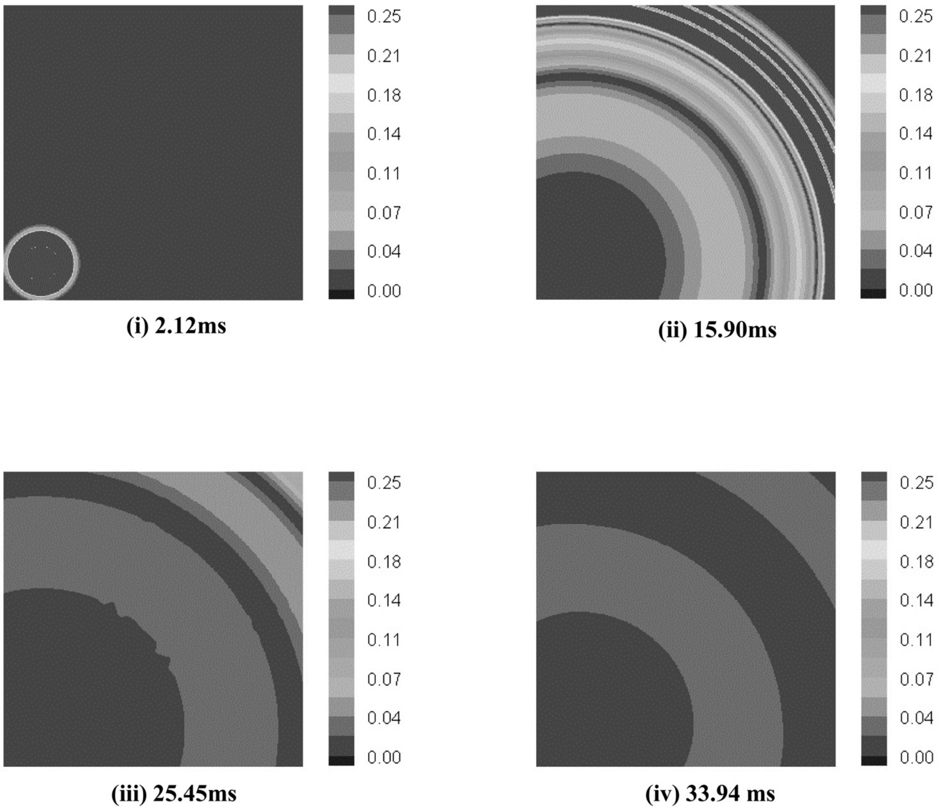

Fig. 4. Padded CFABC solution with three absorbing and three padding layers.

dynamics problems. Eq. (10) is converted to second-order system by using the trapezoidal rule to approximate **X** and **W** successively:

$$\mathbf{X}^{n+1} = \mathbf{X}^{n} + \frac{\Delta t}{2}\left(\mathbf{W}^{n} + \mathbf{W}^{n+1}\right) \tag{8}$$

$$\mathbf{W}^{n+1} = \mathbf{W}^{n} + \frac{\Delta t}{2}\left(\mathbf{U}^{n} + \mathbf{U}^{n+1}\right) \tag{9}$$

The resulting second-order equation is then solved using the standard Newmark schemes.

5. Numerical example

We simulate the propagation of explosion in the unbounded domain modeled using 200×200 mesh of finite elements representing a $30\,\mathrm{m} \times 30\,\mathrm{m}$ interior, with the proposed boundary conditions applied on the boundary. An explosive circular pulse $f(r,t)$ similar to [6] is placed at (3.75m, 3.75m).

$$f(r,t) = \begin{cases} -2\pi^2 f_0^2 (t - t_0) e^{-\pi^2 f_0^2 (t-t_0)^2} \left(1 - \dfrac{r^2}{a^2}\right)^3, \text{ if } t \le 2t_0,\ r \le a \\ 0, \quad \text{otherwise} \end{cases} \tag{10}$$

In the above r is the distance from the center of the source, h is the element length and $a = 5h$ is the radius of the disk, $t_0 = 1/f_0$, $f_0 = c/(hN_L)$ is the central frequency, and N_L is the number of points per wavelength. We used $N_L = 20$ and $c = 2000\,\mathrm{m/s}$. Central difference time-stepping is used and the time step size is taken as the critical value ($\Delta t = h/\sqrt{2}c$). The snapshots of the solution using CFABC with three layers is presented in Fig. 2. Clearly, there is distortion in the wave front, which could be due to the lack of proper treatment of evanescent waves. In order to properly treat evanescent waves, we used three padding layers of thickness 1.27 m, 1.87 m, and 2.74 m respectively (see Fig. 3). The results are shown in Fig. 4, clearly illustrating the effectiveness of padded CFABC.

6. Conclusions

Padded CFABC are new ABCs that constitute special integration coupled with the (real-length) padding and (imaginary-length) absorbers that accurately model evanescent and propagating waves, making themselves highly effective for modeling dispersive exteriors. The implementation of padded CFABC in any existing finite element software is straightforward, with some care required for time-stepping procedure. For further details, the reader is referred to [5].

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