Explicit feedback control for a thermal convection loop

Rafael Vazquez*, Miroslav Krstic

Department of Mechanical and Aerospace Engineering, University of California at San Diego, 9500 Gilman Drive, La Jolla, CA 92093, USA

Abstract

A state feedback boundary control law that stabilizes fluid flow in a 2D thermal convection loop is presented. The fluid is enclosed between two cylinders, heated from above and cooled from below, which makes its motion unstable for a large enough Rayleigh number. This system is widely known for its 'Lorenz system' approximation, being potentially chaotic. The actuation is at the boundary through rotation (direct velocity actuation) and heat flux (heating or cooling) of the outer boundary. The design is a new approach for this kind of coupled PDE problem, based on a combination of singular perturbation theory and the backstepping method for infinite dimensional linear systems. Stability is proved by the Lyapunov method. Though only a linearized version of the plant is considered in the design, an extensive closed loop simulation study of the nonlinear PDE model shows that the result holds for reasonably large initial conditions. A highly accurate approximation to the control law is found in closed form.

Keywords: Nonlinear dynamics; Boundary control; Lyapunov function; Boussinesq approximation; Stabilization; Flow control; Singular perturbations; Backstepping

1. Introduction

A feedback boundary control law is designed for a thermal fluid confined in a closed convection loop, created by heating the lower half of the loop and cooling the upper half. The temperature gradient induces density differences, creating a motion opposed by viscosity and thermal diffusivity. For a large Rayleigh number, the plant develops an instability (and could even go chaotic). The control law is able to suppress this behavior.

Previous efforts include an LQG controller by Burns et al. [1], and a nonlinear backstepping design for a discretized version of the plant [2]. The present design is *simpler* than the former and more *rigorous* than the latter.

Our controller is designed for the linearized plant using a combination of singular perturbation theory [3] and the backstepping method for infinite dimensional linear systems [4]. Combining both methods, a feedback boundary control law is found which stabilizes the closed loop for a large enough Prandtl number, whose inverse plays the role of the singular perturbation parameter.

* Corresponding author. Tel.: +1 (858) 453 1265; Fax: +1 (858) 822 3107; E-mail: rvazquez@ucsd.edu

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2. Problem statement

We employ the model derived in [1], with the geometry shown in Fig. 1; it consists of fluid confined between two concentric cylinders standing in a vertical plane. The main assumptions are a narrow gap between

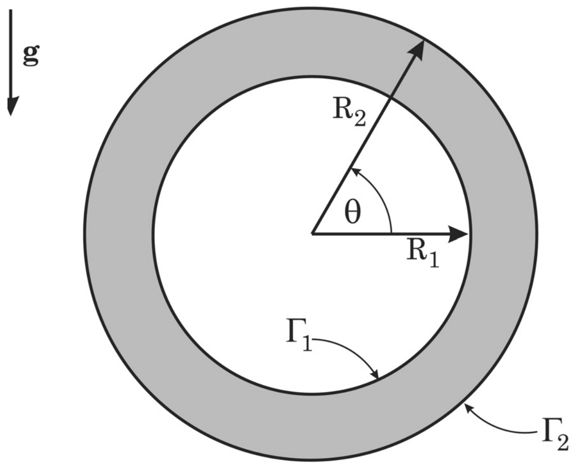

Fig. 1. Convection loop.

the cylinders, i.e. $R_2 - R_1 \ll R_1 < R_2$, and negligible azimuthal velocity. Introducing the Boussinesq approximation, and integrating the momentum equation along circles of fixed radius r, the plant equations are

$$v_t = \frac{\gamma}{2\pi}\int_0^{2\pi} T(t,s,\phi)\cos\ \phi d\phi + \nu\left(-\frac{v}{r^2}+\frac{v_r}{r}+v_{rr}\right) \tag{1}$$

$$T_t = -\frac{v}{r}T_\theta + \mathcal{X}\left(\frac{T_\theta}{r^2}+\frac{T_r}{r}+T_{rr}\right) \tag{2}$$

where v stands for the axial velocity, which only depends, by assumption of the narrow gap, on the radius r; T is the temperature, ν the kinematic viscosity, χ the thermal diffusivity, and $\gamma = g\beta$, with g the acceleration of gravity and β the coefficient of thermal expansion. The assumptions imply absence of the secondary circulation in the model.

The boundary conditions are Dirichlet for velocity, and Neumann in temperature, namely $T_r(t, R_1, \theta) = T_r(t, R_2, \theta) = K\sin\theta$, with K a constant representing the heating and cooling.

Defining $\tau = T - Kr\sin\theta$ we shift the equilibrium to the origin. We also define $r' = r/d$, $t' = t\chi/d^2$, $v' = vd/\chi$, $\tau' = \tau/\triangle T$, $R_a = (1/C)\gamma\triangle d^3/2\nu\chi$, $P = \nu/\chi$, where $d = R2 - R1$, $\triangle T = -(4/\pi)K(R_1 + R_2/2)$, C is a normalizing constant, and R_a and P are respectively the Rayleigh and Prandtl numbers. Then the nondimensional equations are, dropping primes:

$$v_t = \frac{1}{\pi}PR_aC\int_0^{2\pi}\tau(t,s,\phi)\cos\ \phi d\phi + P\left(-\frac{v}{r^2}+\frac{v_r}{r}+v_{rr}\right) \tag{3}$$

$$\tau_t = \frac{d\pi}{2(R_1+R_2)}v\cos\theta - \frac{v}{r}\tau_\theta + \frac{\tau_{\theta\theta}}{r^2}+\frac{\tau_r}{r}+\tau_{rr} \tag{4}$$

The boundary conditions are:

$$v(t,R_1) = 0 \tag{5}$$

$$v(t,R_2) = V(t) \tag{6}$$

$$\tau_r(t,R_1,\theta) = 0 \tag{7}$$

$$\tau_r(t,R_2,\theta) = U(t,\theta) \tag{8}$$

where V and U are the nondimensional velocity and temperature control.

Defining $\epsilon = P^{-1}$, $A_1 = R_aC/\pi$, $A_2 = d\pi/2(R_1 + R_2)$ and linearizing:

$$\epsilon v_t = A_1\int_0^{2\pi}\tau(r,\phi)\cos\ \phi d\phi - \frac{v}{r^2}+\frac{v_r}{r}+v_{rr} \tag{9}$$

$$\tau_t = A_2 v\cos\theta + \frac{\tau_{\theta\theta}}{r^2}+\frac{\tau_r}{r}+\tau_{rr} \tag{10}$$

We will stabilize this linearized plant around its equilibrium at zero.

3. Reduced model

Setting $\epsilon = 0$ and solving Eq. (10) we obtain the quasi-steady-state, which, substituted into Eq. (10), gives the reduced system. Setting the velocity actuation:

$$V = -\frac{A_1}{2}\int_{R_1}^{R_2}\int_0^{2\pi}\frac{R_2^2 - s^2}{R_2}\cos\ \phi\tau(s,\phi)dsd\phi \tag{11}$$

the expression for the quasi-steady-state is

$$v = -\frac{A_1}{2}\int_{R_1}^{r}\int_0^{2\pi}\frac{r^2 - s^2}{r}\cos\ \phi\tau(t,s,\phi)dsd\phi \tag{12}$$

rendering the following reduced system:

$$\tau_t = -A_{12}\int_{R_1}^{r}\int_0^{2\pi}\frac{r^2 - s^2}{r}\cos\ \phi\cos\theta\tau(s,\phi)dsd\phi + \frac{\tau_{\theta\theta}}{r^2} + \frac{\tau_r}{r}+\tau_{rr} \tag{13}$$

where $A_{12} = A_1A_2/2$, with boundary conditions (7)–(8).

4. Backstepping controller for temperature

For stabilization of the reduced system we apply the backstepping technique for parabolic PDEs [4].

The target system is going to be:

$$w_t = \frac{w_{\theta\theta}}{r^2}+\frac{w_r}{r}+w_{rr}, \tag{14}$$

with boundary conditions $w_r(R_1) = 0$, $w_r(R_2) = qw(R_2)$.

For transforming Eq. (13) into Eq. (14) we define:

$$w(r,\theta) = \tau(r,\theta) - \int_{R_1}^{r}\int_0^{2\pi}k(r,\theta,s,\phi)\tau(s,\phi)dsd\phi \tag{15}$$

The kernel k verifies a ultra-hyperbolic PDE, which can be simplified assuming

$$k(r,\theta,s,\phi) = \cos\theta\cos\phi\sqrt{\frac{s}{r}}\hat{k}(r,s) \tag{16}$$

Then $\hat{k}$ verifies a hyperbolic PIDE:

$$\hat{k}_{rr} - \hat{k}_{ss} = \frac{3}{4}\left(\frac{1}{r^2} - \frac{1}{s^2}\right)\hat{k} - A_{12} \left(\frac{r^2 - s^2}{\sqrt{rs}} - \pi \int_s^r \hat{k}(r,\rho)\frac{\rho^2 - s^2}{\sqrt{\rho s}}\,d\rho\right) \quad (17)$$

with boundary conditions:

$$\hat{k}_s(r,R_1) = \frac{\hat{k}(r,R_1)}{2R_1} \quad (18)$$

$$\hat{k}(r,r) = 0 \quad (19)$$

The equation can be solved numerically or reformulated into an integral equation. A first estimate of the kernel is

$$G_0(\xi,\eta) = -A_{12}\Bigg[\frac{1}{6}(\xi^3 - \eta^3 - (\xi^2 - \eta^2)^{3/2} + \frac{5}{2}\sqrt{\pi}R_1^3 e^{1+\eta/R_1} \times \left(\mathrm{erf}(1) - \mathrm{erf}(\sqrt{1+\eta/R_1})\right) + R_1^3(6e^{\eta/R_1} - 34/3) - 8R_1^2\eta - 2R_1\eta^2 + \frac{5}{3}\sqrt{R_1^2 + R_1\eta}\left(5R_1^2 + 2R_1\eta\right)\Bigg] \quad (20)$$

The control law will be

$$U(t,\theta) = q\tau(R_2,\theta) - \cos\theta \int_{R_1}^{R_2}\int_0^{2\pi} \frac{\sqrt{s}\cos\,\phi}{\sqrt{R_2}} \left((q + \frac{1}{2R_2})\hat{k}(R_2,s) - \hat{k}_r(R_2,s)\right)\tau(t,s,\phi)\,ds\,d\phi \quad (21)$$

Introducing the approximation (20), we can get explicit control laws (21) and (11).

5. Singular pertubation analysis for the entire system

Dropping the assumption that $\epsilon = 0$, the following result holds:

Theorem 1 *For a sufficiently small* ϵ, *the system (9)–(10) with boundary conditions (5)–(8), where the actuations V and U are specified by control laws (11) and (21) respectively, has unique classical solutions and is exponentially stable at the origin in the* L^2 *sense.*

6. Simulation study

For numerical computations, a spectral decomposition and a Crank-Nicholson method have been used, using the values $R_1 = 0.365$ m, $R_2 = 0.395$ m, $P = 8.06$, $Ra = 50$, $C = 7.8962 \times 10^3$, $K = 9.113$ C/m. Note that we do not get chaos with these values but on the other hand it is well known that the parameter values that lead to chaos in Lorenz's equations (which are approximated by our plant) are not physical, see for example [5].

Figure 2 is a plot of kernels $\hat{k}(R_2, s)$ and $G_0(R_2 + s, R_2 - s)$, showing an excellent agreement. Figure 3 is an unstable open loop simulation of temperature. In Fig. 4 closed loop simulations are shown, in which the plant is stabilized, along with control effort.

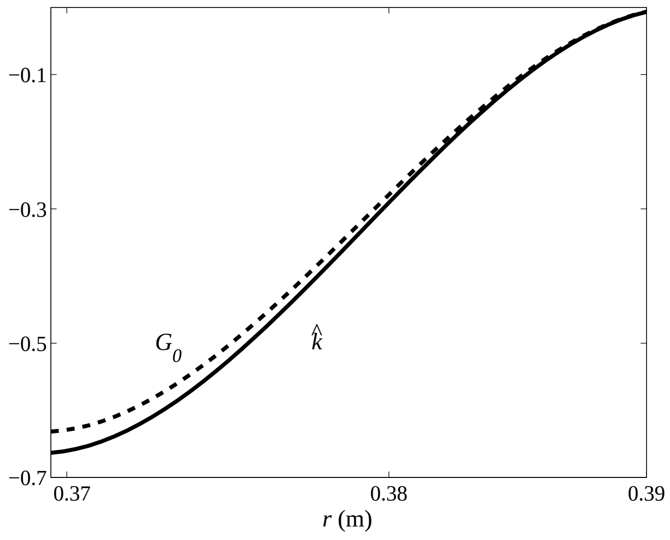

Fig. 2. Exact (solid) and approximate (dashed) control kernels at R_2.

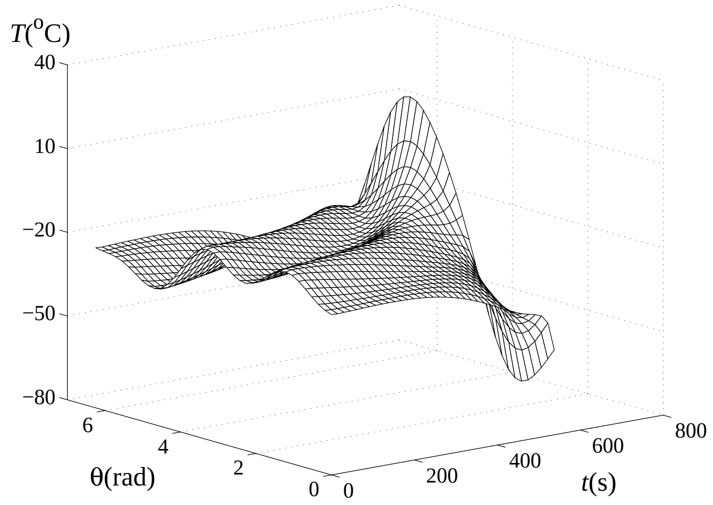

Fig. 3. Open loop evolution of temperature at radius $r = 0.37$ m.

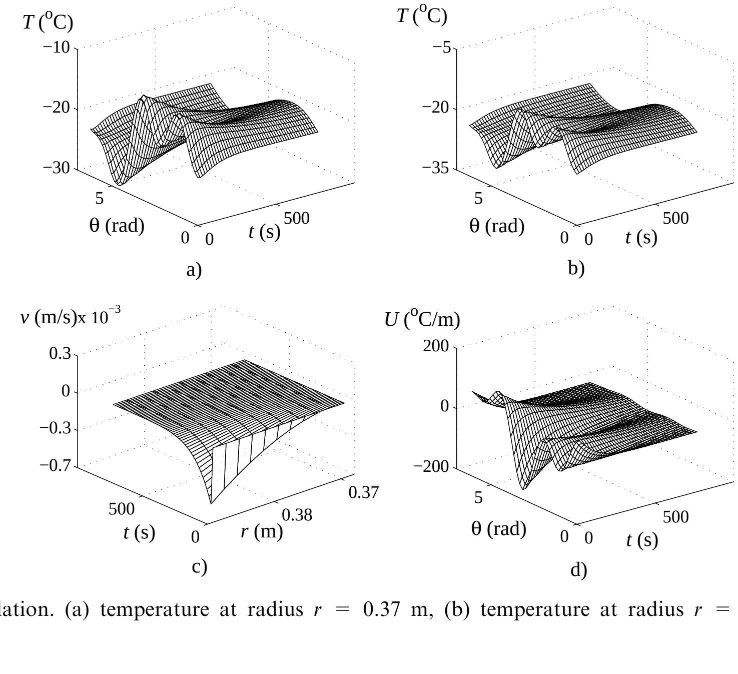

Fig. 4. Closed loop simulation. (a) temperature at radius $r = 0.37$ m, (b) temperature at radius $r = 0.38$ m, (c) velocity, (d) temperature control effort.

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