Incompressible velocity approximations for incompressible fluid flow using DG methods

Dominik Schötzau*

Mathematics Department, University of British Columbia, 1984 Mathematics Road, Vancouver, BC V6T 1Z2, Canada

Abstract

We describe a discontinuous Galerkin method for the incompressible stationary Navier-Stokes equations whose main feature is that it provides a globally divergence-free approximate velocity. This is achieved by a suitable use of a simple, element-by-element post-processing of the completely discontinuous approximations typical of these types of methods. Optimal error estimates are proved and an efficient iterative procedure to compute the approximate solution is shown to converge. Numerical results are displayed that verify the theoretical rates of convergence.

Keywords: Discontinuous Galerkin methods; Incompressible fluid flow

1. Introduction

In this paper, we report on the results obtained in [1]. We show how to solve a difficult issue related to the devising of discontinuous Galerkin (DG) methods for the incompressible Navier-Stokes equations, namely, that locally conservative DG methods are not energy-stable and that energy-stable DG methods are not locally conservative **unless** the approximate velocity is exactly divergence-free. In fact, the current DG methods for the Navier-Stokes equations, namely, [2] and [3] are not locally conservative.

It is well known that it is extremely difficult, if not impossible, to construct finite element spaces for exactly divergence-free velocities. However, we show here that it is possible to avoid having to construct such spaces and still obtain an exactly divergence-free approximate solution. Paradoxically, such an approximation, whose normal components across inter-element boundaries must be continuous, is obtained by taking advantage of the discontinuous nature of the approximations given by DG methods.

In this paper, we sketch the main idea of the construction of such a method, briefly state its main convergence properties and display numerical experiments confirming them.

* Tel.: +1 (604) 822 4346; Fax: +1 (604) 822 6074;
E-mail: schoetzau@math.ubc.ca

Computational Fluid and Solid Mechanics 2005
K.J. Bathe (Editor)

2. The main results

2.1. The idea of the method

The idea of the method is as follows. Given a divergence-free velocity field $\mathbf{w}$ we compute the solution ($\underline{\sigma}$, $\mathbf{u}$, p) of the Oseen equations, namely,

$$\underline{\sigma} = \nu \nabla \mathbf{u}, \quad -\nabla \cdot \underline{\sigma} + (\mathbf{w} \cdot \nabla)\mathbf{u} + \nabla p = \mathbf{f},$$
$$\nabla \cdot \mathbf{u} = 0 \quad \text{in } \Omega \tag{1}$$

We take homogeneous Dirichlet boundary conditions for simplicity.

Let us denote $\mathbf{u}$ by $T(\mathbf{w})$. Then we set $\mathbf{w} := T(\mathbf{w})$ and repeat the process until convergence, that is, until $\mathbf{w} = T(\mathbf{w}) = \mathbf{u}$. In this case, ($\underline{\sigma}$, $\mathbf{u}$, p) is nothing but the solution of the steady state incompressible Navier-Stokes equations.

2.2. The DG method

The DG method mimics this approach. Thus, given an exactly divergence-free velocity field $\mathbf{w}$, we devise an optimally convergent, locally conservative DG method for the Oseen equations. Such a DG method provides an approximation to ($\underline{\sigma}$, $\mathbf{u}$, p) in the space $\underline{\Sigma}_h \times \mathbf{V}_h \times Q_h$ where

$$\underline{\Sigma}_h = \{\mathbf{v} \in L^2(\Omega)^{2\times 2} : \underline{\tau}|_K \in P_k(K)^{2\times 2}, \quad K \in \mathcal{T}_h\}$$
$$\mathbf{V}_h = \{\mathbf{v} \in L^2(\Omega)^2 : \mathbf{v}|_K \in P_k(K)^2, \quad K \in \mathcal{T}_h\}$$
$$Q_h = \{q \in L^2(\Omega) : q|_K \in P_{k-1}(K), \quad K \in \mathcal{T}_h, \int_\Omega q\, d\mathbf{x} = 0\}$$

for an approximation order $k \geq 1$. Here, $P_k(K)$ denotes the space of polynomials of degree at most k. We define the approximate solution $(\underline{\sigma}_h, \mathbf{u}, p_h) \in \underline{\Sigma}_h \times \mathbf{V}_h \times Q_h$ by requesting that for each $K \in \mathcal{T}_h$,

$$\int_K \underline{\sigma}_h : \underline{\tau}\, d\mathbf{x} = -\nu \int_K \mathbf{u}_h \cdot \nabla \cdot \underline{\tau}\, d\mathbf{x} + \nu \int_{\partial K} \hat{\mathbf{u}}_h^\sigma \cdot \underline{\tau} \cdot \mathbf{n}_K\, ds$$
$$\int_K [\underline{\sigma}_h : \nabla \mathbf{v} - p_h \nabla \cdot \mathbf{v}]\, d\mathbf{x} - \int_{\partial K} [\hat{\underline{\sigma}}_h : (\mathbf{v} \otimes \mathbf{n}_K) - \hat{p}_h \mathbf{v} \cdot \mathbf{n}_K]\, ds$$
$$- \int_K \mathbf{u}_h \cdot \nabla \cdot (\mathbf{v} \otimes \mathbf{w})\, d\mathbf{x} + \int_{\partial K} \mathbf{w} \cdot \mathbf{n}_K \hat{\mathbf{u}}_h^{\mathbf{w}} \cdot \mathbf{v}\, ds = \int_K \mathbf{f} \cdot \mathbf{v}\, d\mathbf{x}$$

and

$$- \int_K \mathbf{u}_h \cdot \nabla q\, d\mathbf{x} + \int_{\partial K} \hat{\mathbf{u}}_h^p \cdot \mathbf{n}_K q\, ds = 0 \tag{2}$$

for all test functions $(\underline{\tau}, \mathbf{v}, q) \in \underline{\Sigma}_h \times \mathbf{V}_h \times Q_h$. Each of the above equations is enforced locally, that is, element-by-element, due to the appearance of the so-called *numerical fluxes* $\hat{\mathbf{u}}_h^\sigma, \hat{\underline{\sigma}}_h, \hat{p}_h, \hat{\mathbf{u}}_h^{\mathbf{w}}$ and $\hat{\mathbf{u}}_h^p$. This DG method is both locally conservative as well as energy-stable. See [4] for details.

To be able to iterate as we did for the continuous problem, the crucial point is how to obtain a new exactly divergence-free velocity $\mathbf{w}$ from the completely discontinuous approximations provided by the DG method. This is done by taking $\mathbf{w} := \mathbb{P}\mathbf{u}_h$, where $\mathbb{P}$ is a post-processing operator defined element-by-element by

$$\mathbb{P}\mathbf{u}|_K = \mathbb{P}_K(\mathbf{u}|_K, \hat{\mathbf{u}}^p), \qquad K \in \mathcal{T}_h$$

where $\hat{\mathbf{u}}^p$ is the numerical flux related to the incompressibility constraint. For example, if the elements K are triangles, the local operator $\mathbb{P}_K$ is defined by

$$\int_e \mathbb{P}_K \mathbf{u} \cdot \mathbf{n}_K \varphi\, ds = \int_e \hat{\mathbf{u}}^p \cdot \mathbf{n}_K \varphi\, ds \qquad \forall \varphi \in P_k(e), \text{ for any edge } e \subset \partial K$$
$$\int_K \mathbb{P}_K \mathbf{u} \cdot \nabla \varphi\, d\mathbf{x} = \int_K \mathbf{u} \cdot \nabla \varphi\, d\mathbf{x} \qquad \forall \varphi \in P_{k-1}(K)$$
$$\int_K \mathbb{P}_K \mathbf{u} \cdot \Psi\, d\mathbf{x} = \int_K \mathbf{u} \cdot \Psi\, d\mathbf{x} \qquad \forall \Psi \in \Psi_k(K)$$

where

$$\Psi_k(K) = \{\Psi \in L^2(K)^2 : DF_K^t \Psi \circ F_K \in \Psi_k(\hat{K})\}$$

Here, F_K: $\hat{K} \to K$ denotes the elemental mapping and DF_K its Jacobian. On the reference triangle $\hat{K} = \{(\hat{x}_1, \hat{x}_2) : \hat{x}_1 > 0, \hat{x}_1 + \hat{x}_2 < 1\}$, the space $\Psi_k(\hat{K})$ is defined by

$$\Psi_k(\hat{K}) = \{\Psi \in P_k(\hat{K})^2 : \nabla \cdot \Psi = 0 \text{ in } \hat{K},\ \Psi \cdot \mathbf{n}_{\hat{K}} = 0 \text{ on } \partial\hat{K}\}$$

The post-processing operator $\mathbb{P}$ is well-defined and yields an exactly divergence-free approximation given that $\mathbf{u}_h \in \mathbf{V}_h$ satisfies Eq. (2).

2.3. The convergence results

This iteration has been proven to converge linearly to an approximation of the incompressible Navier-Stokes equations. Moreover, the corresponding solution, whose velocity is exactly incompressible, converges with optimal rates to the exact solution as the mesh-size parameter tends to zero [1].

2.4. Numerical results

We present numerical experiments extracted from [1]. They show that the theoretical rates of convergence are sharp and that the approximate velocity is exactly incompressible. We consider the Kovasznay flow; see [5].

In Table 1 we show the errors and convergence rates in p, $\mathbf{u}$ and $\underline{\sigma}$ obtained for $\nu = 0.1$. The errors in p and $\underline{\sigma}$ are measured in the L^2-norm while $\mathbf{u} - \mathbf{u}_h$ and $\mathbf{u} - \mathbb{P}\mathbf{u}_h$ are evaluated in the classical norm $\|\cdot\|_{1,h}$. We observe the

Table 1
Errors and orders of convergence for $\nu = 0.1$

L	$\|p - p_h\|_0$		$\|\mathbf{u} - \mathbf{u}_h\|_{1,h}$		$\|\mathbf{u} - \mathbb{P}\mathbf{u}_h\|_{1,h}$		$\nu^{-1}\|\underline{\sigma} - \underline{\sigma}_h\|_0$	
3	2.2e+0	–	1.2e+1	–	8.1e+0	–	7.0e−0	–
4	1.0e+0	1.12	5.4e+0	1.11	3.2e+0	1.33	3.4e−0	1.05
5	4.8e−1	1.10	2.4e+0	1.16	1.4e+0	1.18	1.6e−0	1.07
6	2.3e−1	1.04	1.1e+0	1.18	6.8e−1	1.06	7.8e−1	1.04
7	1.2e−1	1.01	4.7e−1	1.17	3.4e−1	1.02	3.9e−1	1.02
8	5.8e−2	1.00	2.2e−1	1.13	1.7e−1	1.01	1.9e−1	1.02

Table 2
L^2-errors and orders of convergence in the velocity and L^∞-norm of the divergence of the post-processed solution $\mathbb{P}\mathbf{u}_h$ for $\nu = 0.1$

L	$\|\mathbf{u} - \mathbf{u}_h\|_0$		$\|\mathbf{u} - \mathbb{P}\mathbf{u}_h\|_0$		$\|\nabla \cdot \mathbb{P}\mathbf{u}_h\|_\infty$
3	6.4e−1	–	4.9e−1	–	1.4e−12
4	1.6e−1	2.03	1.1e−1	2.22	1.4e−12
5	3.3e−2	2.22	2.0e−2	2.37	3.2e−12
6	7.1e−3	2.24	4.2e−3	2.27	1.5e−11
7	1.6e−3	2.19	9.8e−4	2.12	1.8e−12
8	3.5e−4	2.13	2.4e−4	2.04	2.9e−11

predicted first-order convergence for all the error components, in full agreement with the theory.

In Table 2, we display the L^2-errors in the velocities and their corresponding convergence orders. In the first column, we observe that the velocities converge with second order. In the second column, we notice that by post-processing the error is reduced by a factor of roughly 3/2. Therefore, the post-processed solution should be used as the best approximation obtained by our scheme. Furthermore, we show the L^∞-norms of the divergence of $\mathbb{P}\mathbf{u}_h$ (evaluated at the points of a 4-by-4 Gauss formula on each cell). These are of the order of the residual of the non-linear iteration, confirming that the post-processed solution is indeed divergence-free.

3. Concluding remarks

We have shown that, by using DG methods, it is possible to obtain exactly divergence-free approximations of the velocity of the incompressible Navier-Stokes equations. This holds for any polynomial approximation of degree $k \geq 1$. No other finite element method has this capability.

Although the results discussed here have been done in two space dimensions and for triangles, they can be easily extended to other elements, to the three dimensional case and to other incompressible flows.

Extensions of the approach to the Maxwell equations constitute the subject of ongoing work.

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