Computational stability study of 3D flow in a differentially heated 8:1:1 cavity

Andrew G. Salinger*

Computational Sciences Department, Sandia National Laboratories†, Albuquerque, NM 87185, USA

Abstract

The critical Rayleigh number Ra_{cr} of the Hopf bifurcation that signals the limit of steady flows in a differentially heated 8:1:1 cavity is computed. The two-dimensional analog of this problem was the subject of a comprehensive set of benchmark calculations that included the estimation of Ra_{cr} [1]. In this work we begin to answer the question of whether the 2D results carry over into 3D models. For the case of the 2D model being extruded for a depth of 1, and no-slip/no-penetration and adiabatic boundary conditions placed at these walls, the steady flow and destabilizing eigenvectors qualitatively match those from the 2D model. A mesh resolution study extending to a 20-million unknown model shows that the presence of these walls delays the first critical Rayleigh number from 3.06×10^5 to 5.13×10^5.

Keywords: Flow instabilities; Stability analysis; Eigenvalues; Bifurcation; Finite element; Incompressible flow; CFD; Natural convection; Thermal cavity

1. Introduction

The flow in an 8:1 thermally driven cavity was the subject of a comprehensive set of benchmark calculations. At the First MIT Conference on Fluid and Solid Mechanics, 23 contributors presented results for this problem using a variety of numerical methods, as summarized in Christon et al. [1]. Many of these calculations have been formally presented in a special issue of the *International Journal for Numerical Methods in Fluids*, which begins with that article. Part of the challenge was to estimate the location of the Hopf bifurcation signifying the boundary between steady and time-dependent flow solutions. The critical Rayleigh number of this bifurcation was located to very high accuracy by Xin et al. [2] and verified by our group [3]. The delineation in parameter space between steady and unsteady flows was particularly relevant for this problem, since the benchmark was drawn from an application where time-dependent flows would cause unpredictable distortion of a laser passing through the fluid.

This work begins to extend the model to three dimensions, by considering an 8:1:1 cavity where the 2D system has been extruded by a depth of 1.0. No-slip/no-penetration and adiabatic conditions are put on the front and back surfaces. There are three purposes to analyzing the 3D model: to ascertain whether the system loses stability to 3D modes before the 2D mode, to improve on the model for this specific laser application, and to provide a challenging benchmark for 3D flow stability calculations.

2. Problem formulation and solution methods

The detailed description of the 2D problem is contained in Christon et al. [1]. The incompressible Navier-Stokes equations with the Boussinesq approximation along with the continuity and heat equations model the flow in a closed cavity, which is 8 units high and 1 unit wide. There are no-slip/no-penetration boundary conditions around the entire domain, and one side is held at a constant hot temperature and the other side at a cooler temperature. The fluid has a Prandtl number of $Pr = 0.71$, leaving just the Rayleigh number as the free parameter. The definitions of the Rayleigh number (Ra), Prandtl number, and the characteristic time scale are all defined in Christon et al. [1].

* Tel.: +1 (505) 845 3523; E-mail: agsalin@sandia.gov

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In this work, the same model is extruded for a depth of $D = 1$, with no-slip/no-penetration and adiabatic boundary conditions added on these surfaces. The addition of no-slip boundary conditions assures that there will be variation in the solution in the third dimension.

The numerical methods for the PDE discretization, steady solution, and eigenvalue approximation that are used in this work have been previously presented [3,4,5], and involve the MPSalsa, Aztec, LOCA, and ARPACK codes. A hexahedral finite element discretization with trilinear basis functions is used to represent all five variables (u, v, w, P, T) and a pressure stabilization term is added to the Galerkin residual for the continuity equation. A structured mesh was used with grading of the elements towards all surfaces to capture the flow boundary layers. Upwinding (e.g. SUPG) was *not* used in these calculations. A static partitioner was used to spread the domain and work load over a distributed memory parallel computer. A fully coupled Newton method with an analytic Jacobian was used to reach steady-state solutions, and continuation methods were used on the coarsest mesh to reach the elevated values of Ra. The solutions were interpolated from each mesh to the next finest, which provided an adequate initial guess for Newton's method to reconverge on the finer mesh.

The linear system is solved using a parallel domain decomposition ILU preconditioners and a GMRES solver. A generalized Cayley transformation is used to transform the eigenvalue problem so that the rightmost eigenvalues will be among the first to converge within the interative Arnoldi procedure. The Cayley parameters were set at values near $\sigma = -\mu = 1.3$, and 140 Arnoldi iterations were performed without restarts. The Hopf bifurcations were identified by a change in sign of the real part of the rightmost eigenvalues, and the critical Rayleigh number was computed by linear interpolation between two steps in Ra that straddle the bifurcation. The Hopf tracking algorithm in the LOCA library that has been used to directly locate the Hopf point in our 2D calculations [3] was not used because the memory requirement of that algorithm was too large for the problem sizes and available computational resources.

2.1. Stability results for the 8:1:1 cavity

On each of several meshes, the first Hopf bifurcation was located by repeatedly choosing the Rayleigh number, computing a steady solution, and then calculating the leading eigenvalues. Since the real part of the leading eigenvalue, *Real*(λ), was found to vary nearly linearly with Ra, only a few solves were needed to closely bracket the Hopf bifurcation. The approximated eigenspectrum near the Hopf bifurcation ($Ra = 5.15 \times 10^5$ on the finest mesh) is shown in Fig. 1. It can be seen that two modes are on the verge of bifurcating at this point.

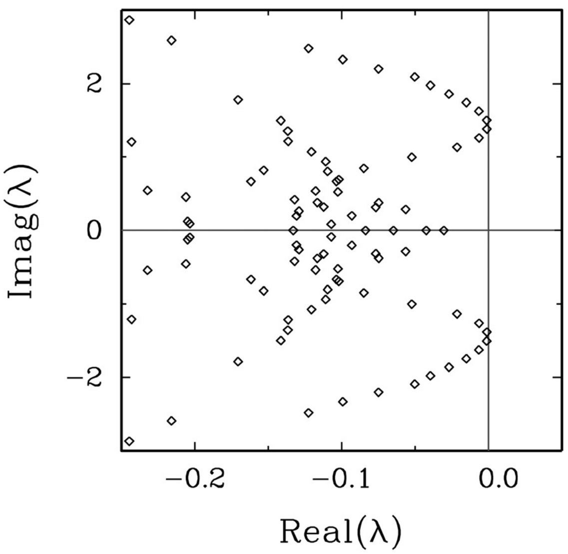

Fig. 1. Part of the eigenspectrum is shown in the vicinity of two Hopf bifurcations, for $Ra = 5.15 \times 10^5$ on the 20-million unknown mesh. Note that the x-axis has been stretched by a factor of 20 since these eigenvalues identified after use of the Cayley transformation are all near the imaginary axis. Only the four rightmost pairs of eigenvalues are converged to several digits.

Visualization of the solution and real part of the destabilizing eigenvector are shown in Fig. 2. Visual comparison of these solutions with those for the 2D problem (e.g. Fig. 2 in Salinger et al. [3]) shows that the flow and eigenmode are closely related to those for the 2D problem. While the number of vortices in the eigenvector may not be exactly the same, it appears to be the same phenomena that destabilizes both the 2D model and this 3D model.

The leading critical Rayleigh number was identified for a series of meshes, ranging from 330 thousand to over 20 million unknowns. The finest mesh consisted of a $320 \times 128 \times 96$ discretization in the height, width, and depth respectively. All meshes preserved this element ratio between the spatial dimensions. Table 1 shows the results of these calculations, reporting the leading two critical Rayleigh numbers and the frequencies ω at the Hopf bifurcations, along with the CPU time required to perform an eigenvalue calculation at these conditions. (The computer contains 3.06 GHz Intel Xenon processors with 2 Gb of memory for each 2-processor node.) Note that the gap between the first and second critical Rayleigh numbers has almost disappeared on the finest mesh.

By assuming the order h^2 convergence of the critical

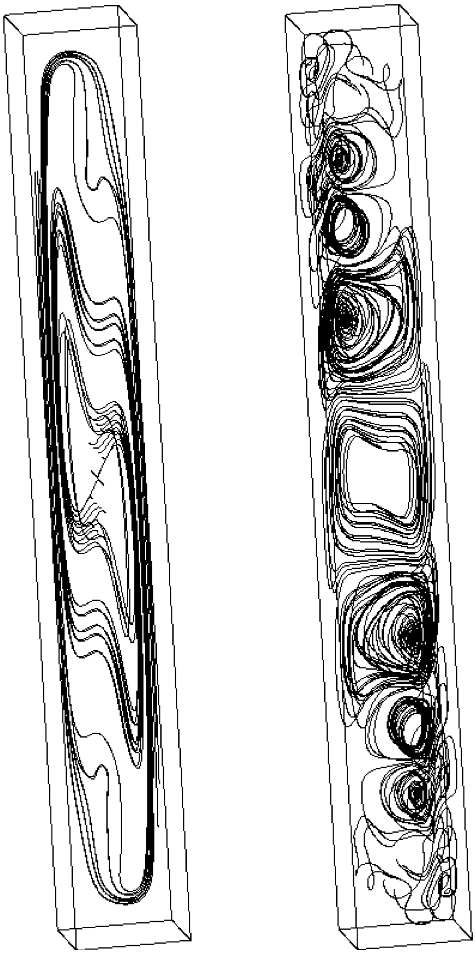

Fig. 2. Streamlines are shown for the steady flow (left) and the destabilizing eigenvector (right) near the instability.

Rayleigh number that we have seen in previous computations [3,4], we have extrapolated the results to estimate the true solution of the PDEs. The results of these extrapolations are shown in the final row of Table 1. One interesting result is that the extrapolation predicts that the second critical Rayleigh number overtakes the first (5.134×10^5 *vs.* 5.137×10^5). A plot of the first critical Rayleigh number with respect to mesh spacing h is shown in Fig. 3 along with the quadratic extrapolation as $h \rightarrow 0$. For these calculations, we defined the mesh spacing $h = (N_n)^{-1/3}$ for number of nodes N_n. (The data for the second critical Rayleigh number nearly falls on top of this data, and is not shown.) As can be seen on this plot, there are not enough data points that fall on the quadratic fit to conclusively predict the critical Rayleigh number to four decimal points, or even to conclusively say whether the mode with the higher or lower frequency bifurcates first in the limit of $h \rightarrow 0$. It is expected that both bifurcations occur in the range of $5.13 \pm 0.03 \times 10^5$.

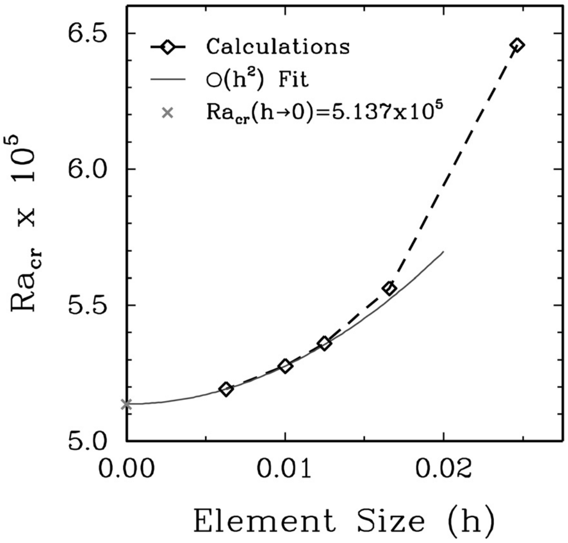

Fig. 3. The critical Rayleigh number Ra_{cr} calculated on five successively refined meshes is plotted versus mesh spacing h. Extrapolation assuming quadratic convergence gives a prediction of the true Ra_{cr} for the PDEs.

Table 1
The results of the mesh convergence study showing the dependence of the first two critical Rayleigh numbers on mesh are shown. The columns represent number of unknowns in the mesh, the values of Ra_{cr} and the frequency for each of the first two Hopf bifurcations, and the number of CPU hours (and number of processors) used to approximate the eigenvalues on this mesh

Unknowns	${Ra_{cr}}^{(1)}$	$\omega^{(1)}$	${Ra_{cr}}^{(2)}$	$\omega^{(2)}$	CPU hrs (Procs)
0.334M	6.457×10^5	1.359	–	–	0.5 (20)
1.097M	5.563×10^5	1.372	5.630×10^5	1.493	1.0 (40)
2.564M	5.361×10^5	1.377	5.379×10^5	1.498	2.1 (64)
4.966M	5.277×10^5	1.378	5.283×10^5	1.500	4.4 (80)
20.083M	5.192×10^5	1.379	5.193×10^5	1.501	16.0 (240)
∞	5.137×10^5	1.380	5.134×10^5	1.502	(Extrapolation)

Table 2
The sensitivity of Ra_{cr} and ω on the depth of the cavity is estimated by finite differencing the results at $D = 1.0$ with that at $D = 0.999$ for the 2.564M unknown mesh

Box depth (D)	Ra_{cr}	ω	$\frac{dRa_{cr}}{dD}$	$\frac{d\omega}{dD}$
1.0	5.3609×10^5	1.37669		
0.999	5.3705×10^5	1.37656	-1.04×10^6	0.13

One final calculation was performed to aid in extrapolating these results to other box depths D. Table 2 shows the results of recalculating the critical Rayleigh number at $D = 0.999$, and the finite difference approximation of $\mathrm{dRa_{cr}/dD}$ at $D = 1$. Since the 2D calculations represent the $D \rightarrow \infty$ result, we have enough data to set the parameters in a power law fit to the $Ra_{cr}(D)$ function.

$$Ra_{cr} = 3.06 \times 10^5 + 2.07 \times 10^5 (D)^{-n} \quad (1)$$

where the derivative result leads to an estimate near $n = 5$. More data is needed to see if this correlation has any merit.

3. Conclusions

A linear stability analysis has been performed on the flow in a differentially heated 8:1:1 cavity, and compared to the results of 2D calculations. It is found that the flow and destabilizing eigenmode remain qualitatively the same in the 3D case. The addition of the no-slip walls in the third dimension have a stabilizing effect on the flow, delaying the critical Rayleigh number from 3.06×10^5 for the 2D case to near 5.13×10^5 for the 3D case. Our calculations show that the first two critical Rayleigh numbers, corresponding to two Hopf bifurcations, occur at nearly the same point. It is not clear from our calculations which one would bifurcate first as the mesh spacing was dropped to zero.

The estimation of the critical Rayleigh number for this system involved a mesh resolution study that included eigenvalue approximations for a system of over 20 million unknowns, which is the largest ever performed by this research group.

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