A modified conservation law approach to improved finite element incompressible Navier–Stokes algorithms

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Abstract

Higher-order Navier–Stokes algorithms are the need of the times to harness emerging computing power. The Taylor Weak Statement (TWS) modified conservation law approach, coupled with higher-degree-basis finite element implementation, is a unique way to accomplish this goal. The TWS approach to improved accuracy conservation law algorithm solutions is presented.

Keywords: CFD algorithm; Modified equations; Phase accuracy

1. Introduction

The Galerkin weak statement applied to Navier–Stokes (NS) conservation law systems is intrinsically unstable for large Reynolds number applications. Perhaps the most common resolution has been to alter the test space, and hence switch to a non-Galerkin weak form, to introduce numerical diffusion to stabilize the solution process. One must be extremely careful here, however, as this can lead to a change in the physics of the problem itself by reducing the effective Reynolds number of the simulation. An alternative very systematic approach to generation of stable and accurate algorithms, termed the Taylor Weak Statement (TWS) process is presented herein.

2. Problem statement

2.1. TWS background

Donea [1] pioneered the original Taylor-Galerkin algorithm and applied it to convective transport problems. Baker et al. [2] generalized the concept in developing the TWS-modified hyperbolic conservation law process and documented improved performance over GWS algorithm solutions. Chaffin et al. [3] applied TWS to incompressible Navier-Stokes (INS) systems via a linear FE basis implementation. Kolesnikov et al. [4] generalized it for the steady INS system generating an improved accuracy linear basis algorithm. TWS-categorizes reported CFD algorithms as organized via choices for available parameters [2] in Table 1.

Current research focus is on higher-degree finite element basis implementations of TWS for unsteady INS. The Taylor series analysis framework yields a modified continuum restatement of the Navier-Stokes conservation law system yielding the opportunity for decisive analysis on optimal parameter selection. This approach results in a genuine Galerkin weak statement, i.e. the error is orthogonal to the trial space.

2.2. Theoretical analysis

The INS conservation law statement is

$$L(u_i) = \frac{\partial u_i}{\partial t} + \frac{\partial}{\partial x_j}\left(u_i u_j + \frac{p}{\rho}\delta_{ij} - \sigma_{ij}\right) + b_i = 0 \qquad (1)$$

$$\frac{\partial u_i}{\partial x_i} = 0 \qquad (2)$$

The Taylor series modified INS statement, Eq. (1), with α, β, γ, and μ arbitrary parameters is

$$L^m(u_i) = L(u_i) - \frac{\Delta t}{2}\frac{\partial}{\partial x_j}\left(\alpha u_j \frac{\partial u}{\partial t} + \beta u_j u_k \frac{\partial u}{\partial x_k}\right)$$

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Table 1
TWS algorithm categorization of reported CFD algorithms [1]

Algorithm name	θ	α	β	γ	μ
$TWS^h + \theta TS$	all	arbitrary	arbitrary	arbitrary	arbitrary
(Bubnov)GWS^h	all	0	0	0	0
Donor cell FD	0	0	1	0	0
Lax-Wendroff FD	0	0	sgn(u)	0	0
Euler Taylor GWS^h	0	0	1	1	0
CN Taylor GWS^h	0.5	0	0.5	1	0
Euler Char. GWS^h	0	0	1	0	1
Swansea Taylor GWS^h	0	0	1	0	0
Wahlbin	0	sgn(u)	2sgn(u)	0	0
Dendy	0	$\Delta x \cdot$sgn(u)	$\Delta x \cdot$sgn(u)	0	0
Raynond-Garder	0.5	sgn(u)/$v_\circ$	sgn(u)/v_o	0	0
Hughes SUPG	–	0	sgn(u)	0	0
Euler Petrov GWS^h	0	0	0	$(1 - v)$	0
CN Petrov GWS^h	0.5	sgn(u)	v·sgn(u)	$-v/2$	0
Warmin-Beam FD	0	0	1	0	$-3(1 - C)$
VanLeer MUSCL	1	0	sgn(u)	0	-3
Jiang Least Squares	all	2θ	2θ	0	0

Note sgn(u) is the sign of u, $v_\circ = 1/\sqrt{15}$, $C \leq v \leq 1$

$$-\frac{\Delta t^2}{6}\frac{\partial}{\partial x_j}\left[\gamma u_j u_k \frac{\partial}{\partial x_k}\frac{\partial u}{\partial t} + \mu u_j u_k \frac{\partial}{\partial x_k}\left(u_l \frac{\partial u}{\partial x_l}\right)\right] \tag{3}$$

where ρ is the (constant) density, p is the pressure, b_i is the body force, σ_{ij} is the Stokes stress tensor, and u_i is the velocity vector. Solutions to Eq. (3) are parameterized by the Reynolds number, Re, which is of order $Re \geq 10^4$ for practical problem statements.

Hence, theoretical attention is focused on the pure advection limiting form $Re^{-1} \to 0$ of Eq. (3). In one dimension, the associated modified transport equation for generic variable q(x,t) is

$$L^m(q) = \frac{\partial q}{\partial t} + u\frac{\partial q}{\partial x} - \frac{\Delta t}{2}\frac{\partial}{\partial x}\left(\alpha u \frac{\partial q}{\partial t} + \beta u^2 \frac{\partial q}{\partial x}\right) \tag{4}$$

$$-\frac{\Delta t^2}{6}\frac{\partial}{\partial x}\left[\gamma u^2 \frac{\partial}{\partial x}\frac{\partial q}{\partial t} + \mu u^2 \frac{\partial}{\partial x}\left(u \frac{\partial q}{\partial x}\right)\right] = 0$$

The resultant TWS algorithm solution amplification factor for linear basis implementation is

$$G^h = \frac{(2 + \gamma C^2 - 3(1-\theta)\beta C^2) + \cos\sigma(1 - \gamma C^2 + 3(1-\theta)\beta C^2) + i \sin\sigma\left(\frac{-3}{2}\alpha C - 3(1-\theta)C\right)}{(2 + \gamma C^2 + 3\theta\beta C^2) + \cos\sigma(1 - \gamma C^2 - 3\theta\beta C^2) - i\sin\sigma\left(\frac{3}{2}\alpha C - 3\theta C\right)} \tag{5}$$

Where θ and C are an implicit parameter and the cournant number, respectively. Attention is focused on the impact of the γ parameter on the phase accuracy of the TWS algorithm solution. The reduced form is

$$L^m(q) = \frac{\partial q}{\partial t} + u\frac{\partial q}{\partial x} - \frac{\gamma \Delta t^2 u^2}{6}\frac{\partial}{\partial x}\left(\frac{\partial^2 q}{\partial x \partial t}\right) \tag{6}$$

with the associated linear basis amplification factor

$$G^h = \frac{(2 + \gamma C^2) + \cos\sigma(1 - \gamma C^2) - i\sin\sigma(3(1-\theta)C)}{(2 + \gamma C^2) + \cos\sigma(1 - \gamma C^2) + i\sin\sigma(3\theta C)} \tag{7}$$

Equation (7) clearly indicates that G^h is the ratio of complex conjugates only for $\theta = \frac{1}{2}$. Then, the amplification factor magnitude is unity for all γ, i.e. no artificial dissipation is present. The associated phase velocity is

$$\Phi^h = \frac{1}{-\sigma C}\tan^{-1}\left(\frac{\Im(G^h)}{\Re(G^h)}\right) \tag{8}$$

from which relative phase propagation error distribution for the wave numbers $0 \leq \sigma \leq \pi$ can be determined.

Determining existence of optimal γ involves writing a Taylor expansion of G^h, and comparing to the analytical amplification factor, $G = \exp(-i\sigma C)$, as a function of wave number. The process can be repeated for TWS algorithm implementation using quadratic and cubic FE bases.

3. Discussion and results

A TWS algorithm solution for Eq. (6) is time accurate, so the parametric study involves the Courant number ($C = u \cdot \Delta t / \Delta x$) and γ. The results show an interesting interdependence between C and γ. For a smooth initial condition, the quadratic FE basis TWS solution is stable for a range of γ but becomes immediately unstable when the range is exceeded. The stability range, hence the optimal γ parameter, appears

Table 2
Optimal γ for C and FE basis degree, k

Basis degree	C = 0.25	C = 0.5	C = 1.0
k = 1	−0.7	−0.6	−0.5
k = 2	–	−0.4	−0.4
k = 3	–	−0.25	−0.33

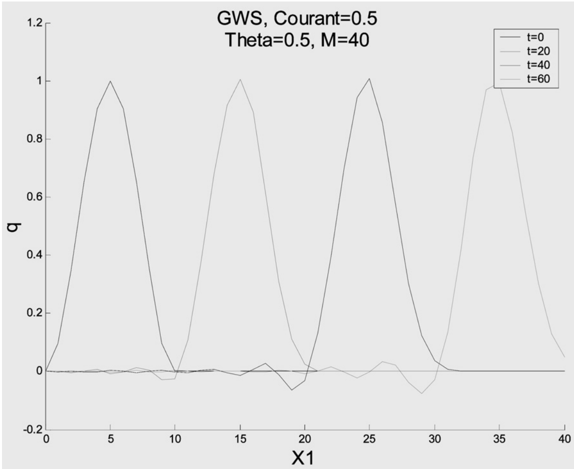

Fig. 1. Unsteady scalar field transport, GWS, C = 0.5, θ = 0.5, k = 1, M = 40.

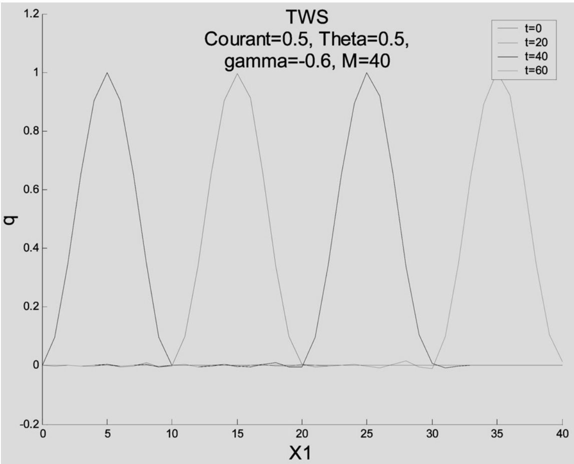

Fig. 2. Unsteady scalar field transport, TWS, C = 0.5, θ = 0.5, k = 1, γ = −0.6, M = 40.

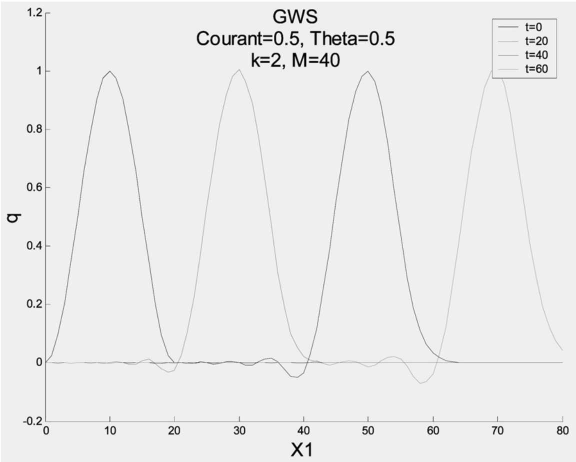

Fig. 3. Unsteady scalar field transport, GWS, C = 0.5, θ = 0.5, k = 2, M = 40.

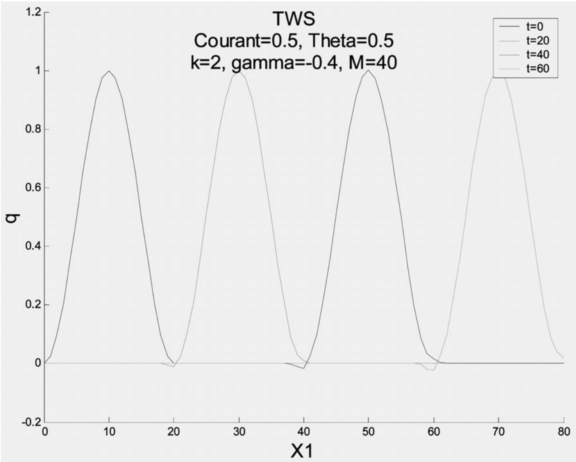

Fig. 4. Unsteady scalar field transport, TWS, C = 0.5, θ = 0.5, k = 2, γ = −0.4, M = 40.

unique for linear, quadratic and cubic FE bases. For example, assembling TWS algorithm at generic node 'j', for C = 1 and using the linear basis FE implementation produces the recursion relation

$$\left(Q_j + Q_{j+1}\right)_{n+1} = \left(Q_j + Q_{j-1}\right)_n \tag{9}$$

which confirms that all nodal data are propagated exactly. Table 2 summarizes basis-degree dependent 'optimal' γ as a function of C as determined numerically.

Figures 1 and 2 graph the time evolution of the k = 1 basis GWS and TWS solutions for C = 0.5. The optimal solution results for γ = −0.6. Figures 3 and 4 compare the time evolution of the k = 2 basis GWS and TWS

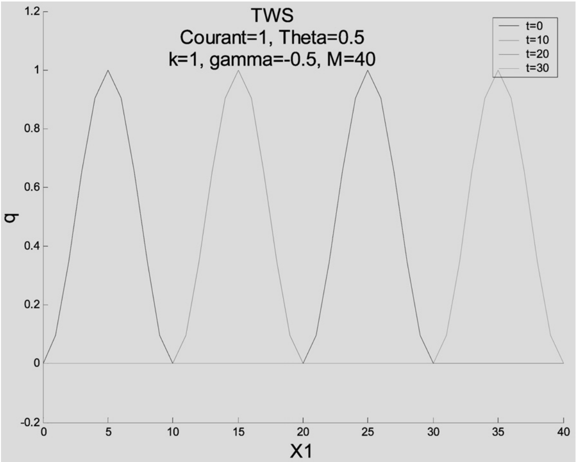

Fig. 5. Unsteady scalar field transport, TWS, C = 1.0, θ = 0.5, k = 1, γ = −0.5, M = 40.

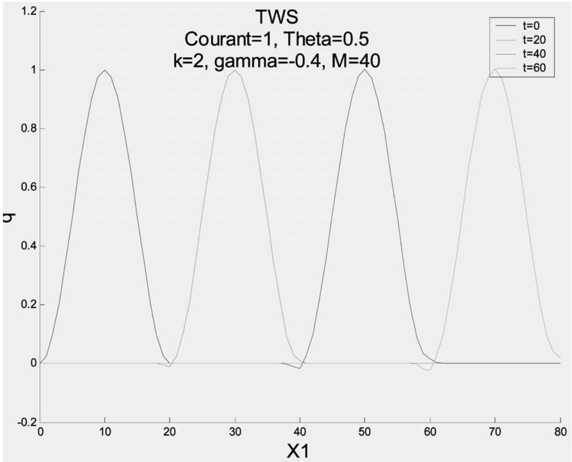

Fig. 6. Unsteady scalar field transport, TWS, C = 1.0, θ = 0.5, k = 2, γ = −0.4, M = 40.

solutions for C = 0.5. Now the optimal solution results for $\gamma = -0.4$. Figures 5 and 6 compare the time evolution of TWS solutions for C = 1, for k = 1 and 2 bases respectively. The optimal γ is −0.5 and −0.4, respectively, and interestingly the singular C = 1, k = 1 basis solution is more phase accurate than the k = 2 solution.

4. Conclusions

The INS TWS-modified conservation law form is summarized, indicating the subsequent Fourier modal analysis for performance comparisons. Improved phase accurate results are verified for a simple 1D problem. TWS theory provides a theoretical framework for potential improvement of CFD algorithms in n-dimensions via select order FE basis implementations on arbitrary meshes.

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