Numerical approximation of the spectra of Poiseuille flow of two Phan-Thien Tanner liquids

A.S. Palmer*, T.N. Phillips

School of Mathematics, Cardiff University, Cardiff CF24 4YH, UK

Abstract

The eigenspectrum for the planar Poiseuille flow of two immiscible Phan-Thien Tanner liquids is studied. The numerical method is to discretise spectrally employing the Chebyshev-tau method and the full set of eigenvalues are determined by the QZ-algorithm. The effects of variations in the fluid parameters are investigated and comparisons are made between the two-fluid and single fluid cases. The flow is found to be stable for the ranges of parameters investigated so far.

Keywords: Phan-Thien Tanner liquids; Visco-elastic liquids; Poiseuille flow; Two-layer flow; Chebyshev; Eigenspectrum; Linear stability analysis

1. Introduction

Many products are derived through the extrusion or coextrusion of polymer melts and polymer solutions such as multilayered films and coating. Often it is desirable for the final product to be smooth and undistorted so as to maximise the quality and use of the product. Investigations of the stability of these fluid flows can lead to enlightenment as regards optimal choices within the manufacturing process.

Much work has been done on the stability of Poiseuille flows of Oldroyd-B and UCM fluids, for example Wilson et al. [1] and Ganpule et al. [2], and has shown few instabilities. A recent stability analysis of the Poiseuille flow of the exponential form of the PTT model by Grillet et al. [3] has predicted the onset of instabilities within certain ranges of the model parameters. The work described in this paper considers a linear stability analysis of the linear PTT model to determine whether or not a similar conclusion holds for this version of the PTT model. The results given by Palmer et al. [4] describe the eigenspectra for the Poiseuille flow of a single linear PTT fluid and this work is extended here to the case of two immiscible fluids within the same channel.

* Corresponding author. Tel.: +44 (1970) 622771; Fax: +44 (1970) 622826; E-mail: PalmerA2@cardiff.ac.uk

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2. Formulation

The problem under consideration is that of planar Poiseuille flow of two immiscible Phan-Thien Tanner fluids. The geometry is two-dimensional with the x-direction being the direction of the flow. Fluid 1 occupies the lower layer within the channel, $y \in [0, l]$, and fluid 2 occupies the upper layer, $y \in [l, 1]$.

The Phan-Thien Tanner models [5,6] for polymer melts and solutions are derived from a Lodge-Yamamoto type of network theory [7,8,9,10] whereby the polymer molecules are represented as chains with junctions at the points where molecules meet. The junctions can be created and destroyed as the fluid moves and are also allowed to slip along the chains such that the strand stretching and total motion of the fluid are not necessarily related by an affine transformation.

The nondimensional governing equations to be used are the conservation of momentum, continuity and the Phan-Thien Tanner constitutive equations, respectively:

$$R_j\left(\frac{\partial \mathbf{v}_j}{\partial t} + (\mathbf{v}_j \cdot \nabla)\,\mathbf{v}_j\right) = -\nabla p + \nabla \cdot \boldsymbol{\tau}_j + m_j\beta_j\nabla^2\mathbf{v}_j \qquad (1)$$

$$\nabla \cdot \mathbf{v}_j = 0 \qquad (2)$$

$$W_j\overset{\square}{\boldsymbol{\tau}}_j + \frac{\epsilon_j W_j}{m_j(1-\beta_j)}\mathrm{tr}(\boldsymbol{\tau}_j)\boldsymbol{\tau}_j + \boldsymbol{\tau}_j = 2m_j(1-\beta_j)\,\mathbf{d}_j \qquad (3)$$

Here the general convective (Johnson-Segalman) derivative of the stress tensors is

$$\overset{\square}{\boldsymbol{\tau}} = \frac{\partial \boldsymbol{\tau}}{\partial t} + (\mathbf{v} \cdot \nabla)\boldsymbol{\tau} - \mathcal{L}\boldsymbol{\tau} - \boldsymbol{\tau}\mathcal{L}^{\mathrm{T}} \tag{4}$$

the relationship between strand-stretching and total fluid motion is

$$\mathcal{L} = \nabla \mathbf{v} - \xi \mathbf{d} \tag{5}$$

and the rate of strain is given by

$$\mathbf{d} = \frac{1}{2}(\nabla \mathbf{v} + (\nabla \mathbf{v})^{\mathrm{T}}) \tag{6}$$

The subscript $j = 1, 2$ denotes the fluid to which the equations correspond. The vectors $\mathbf{v}_j$ denote the velocities, p_j denote pressures and $\boldsymbol{\tau}_j$ denote the stress tensors. There are four dimensionless parameters for each fluid; the Reynolds numbers R_j, the Weissenberg numbers W_j (a measure of the elasticities of the fluids), the solvent to total viscosity ratios $\beta_{\mathrm{j}} = \frac{\eta_{s_{\mathrm{j}}}}{\eta_{t_{\mathrm{j}}}}$ and the comparative total viscosity ratios $m_{\mathrm{j}} = \frac{\eta_{t_{\mathrm{j}}}}{\eta_{t_1}}$. The adjustable parameters of the PTT fluid are the extensional parameter ξ and the shear-thinning parameter ϵ. The PTT fluid model reduces to the Oldroyd-B model for polymer solutions when $\epsilon = \xi = 0$ and further reduces to the UCM model for polymer melts when $\beta = 0$.

For the solution of the base flow it is assumed that the flow is one-dimensional and hence dependent on y only. The boundary conditions are no slip at the channel walls,

$$(u_{\mathrm{B}})_1\big|_{y=0} = 0 \tag{7}$$

and

$$(u_{\mathrm{B}})_2\big|_{y=1} = 0 \tag{8}$$

The presence of the interface introduces the requirement for expressions of continuity of velocity and traction across this interface.

The linear stability analysis and the computation of the eigenspectrum is performed in the usual manner by considering small perturbations to the base flow in the form $\bar{\mathrm{u}}e^{i\alpha x+\sigma t}$ where σ is the eigenvalue and the $\bar{\mathrm{u}}$ are of infinitesimally small magnitude such that quadratic and higher-order terms can be neglected.

3. Numerical method

Neither the base flow nor the eigenspectrum has been fully calculated analytically so numerical solutions are sought. A spectral method is followed whereby the equations are discretised using the Chebyshev-tau method. All flow variables are approximated by expansions in terms of Chebyshev polynomials. The infinite sequence of Chebyshev polynomials is an exact representation of the flow variables but for computational purposes it is necessary to take an approximation to the variables as

$$z_N = \sum_{n=0}^{N} [\hat{z}]_n [T]_n \tag{9}$$

where N is suitably large. Thus the accuracy of the numerical solution is governed by the chosen value of N.

Discretising in this manner leads to the vector equation

$$(\mathbf{T}_{\mathrm{B}})^{\mathrm{T}} \cdot \mathbf{F}_{\mathrm{B}} \cdot \mathbf{z}_{\mathrm{B}} = 0 \tag{10}$$

for the base flow and

$$(\mathbf{T}_{\mathrm{LSE}})^{\mathrm{T}} \cdot \mathbf{A}_{\mathrm{LSE}} \cdot \mathrm{z}_{\mathrm{LSE}} = \sigma\, (\mathbf{T}_{\mathrm{LSE}})^{\mathrm{T}} \cdot \mathbf{B}_{\mathrm{LSE}} \cdot \mathbf{z}_{\mathrm{LSE}} \tag{11}$$

for the linearised stability equations (LSEs). The vectors $\mathbf{T}_{\mathrm{B}}$ and $\mathbf{T}_{\mathrm{LSE}}$ contain the Chebyshev polynomials, the vectors $\mathbf{z}_{\mathrm{B}}$ and $\mathbf{z}_{\mathrm{LSE}}$ contain the Chebyshev coefficients and the matrices $\mathbf{F}_{\mathrm{B}}$, $\mathbf{A}_{\mathrm{LSE}}$ and $\mathbf{B}_{\mathrm{LSE}}$ contain coefficients expressing the relationships between the Chebyshev polynomials and coefficients as determined by the governing equations.

Given a satisfactory initial guess for the values of the Chebyshev coefficients taken from the analytical solutions for the Oldroyd-B model, Newton iteration is used to find a discrete representation of the base flow within a given tolerance.

In order to find the complete eigenspectrum the base flow results are substituted into the LSEs and the entries of the matrices $\mathbf{A}_{\mathrm{LSE}}$ and $\mathbf{B}_{\mathrm{LSE}}$ are used by a *NAG* routine to compute the eigenvalues of the system using the QZ-algorithm [11]. Since the base flow results are required for the solution of the LSEs the same value of N is used in the discretisations of the base flow equations and the LSEs.

4. Results

The main results are shown in Figs 1, 2 and 3. For the two-fluid problem the eigenspectrum consists of two eigenspectra overlying one another and these are very similar to those found for the single fluid problem but with the addition of a marginally stable interfacial eigenvalue. The eigenspectra for the single fluid flows consist of two parts, which are referred to as the 'Oldroyd-B' part and the 'UCM' part in the literature. All results here are for $W = 1$, $R = 0$, $\beta = 0.2$, $\alpha = 1$ and with the interface at the midpoint of the channel. For these parameter values the UCM eigenspectrum lies near

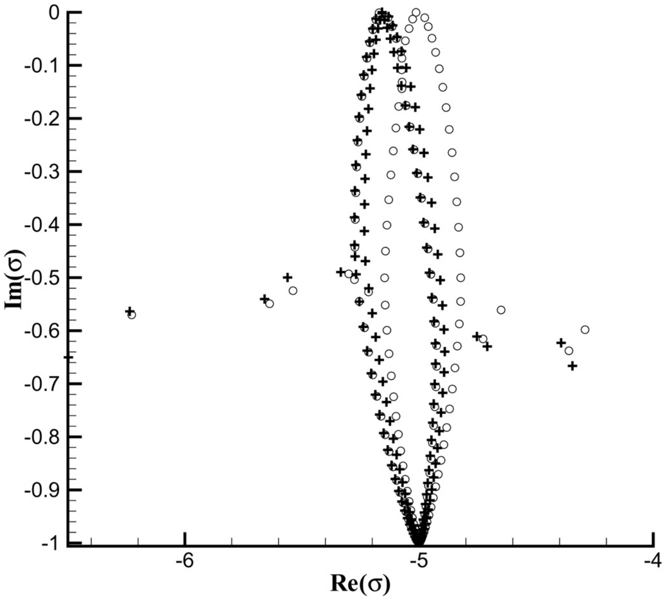

Fig. 1. The Oldroyd-B part of the eigenspectra for the parameter values $\xi_1 = \xi_2 = 0$, $\epsilon_1 = \epsilon_2 = 0.05$ (+) and $\epsilon_1 = 0$, $\epsilon_2 = 0.05$ (○).

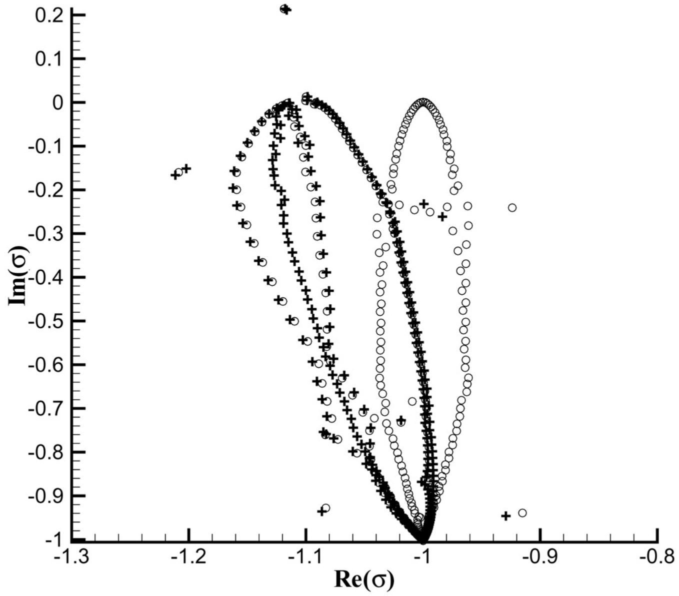

Fig. 2. The UCM part of the eigenspectra for the parameter values $\xi_1 = \xi_2 = 0$, $\epsilon_1 = \epsilon_2 = 0.05$ (+) and $\epsilon_1 = 0$, $\epsilon_2 = 0.05$ (○).

to $\Re(\sigma) = -1 = -^1\!/_{W}$ and the Oldroyd-B part near to $\Re(\sigma) = -5 = -^1\!/_{W\beta}$. Both of these parts of the eigenspectrum consist of a continuous part with a number of discrete eigenvalues within close proximity. The continuous spectra reveal themselves as 'balloons' of eigenvalues and these balloons become narrower as the number of Chebyshev modes is increased, thus increasing the resolution of the results. All results given in this paper are for $N = 90$.

Figures 1 and 2 show the effect of altering the shear-thinning parameters ϵ_j. The pluses show the eigenspectrum for the case when the fluid parameter values are the same in each fluid. This is equivalent to the single fluid problem and the results correspond with earlier results

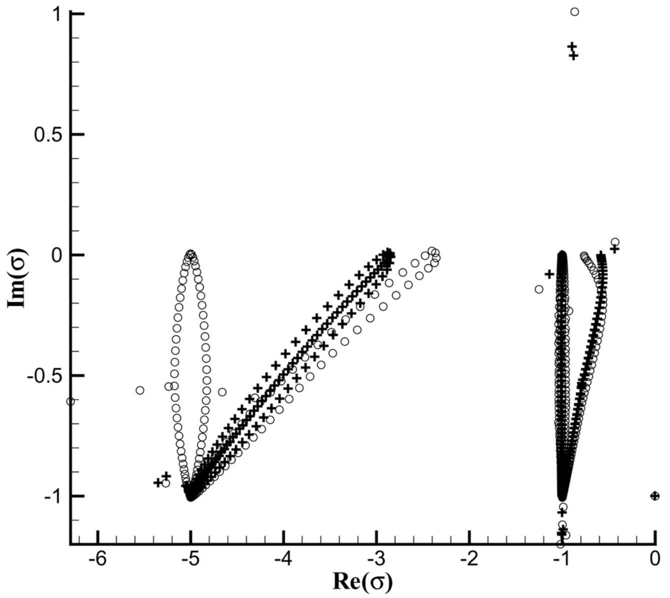

Fig. 3. The complete eigenspectra for the parameter values $\varepsilon_1 = \varepsilon_2 = 0$, $xi_1 = xi_2 = 0.5(+)$ and xi_1, $xi_2 = 0.5$ (∘).

for the single fluid [4]. The tilt of the eigenspectra towards the negative reals increases as ϵ is increased, which would imply that, in this case, shear-thinning has a stabilising effect. The circles show the eigenspectrum when the shear-thinning parameter is nonzero in only one fluid. It is of no consequence which fluid is given the higher value of ϵ since there is no consideration of gravity in this problem. It is clear that there are two eigenspectra overlying each other, one being very similar to that found for a single Oldroyd-B fluid [1] and the other being similar to that found for a single fluid with $\epsilon = 0.05$. There is some interaction between the two eigenspectra however and this would be expected since the fluids themselves will interact at the interface. This interaction between the fluids reveals itself by the differences between the two eigenspectra for the two-fluid flow and the single fluid equivalents. The main trend however, that the introduction of shear-thinning has a stabilising effect, is unchanged. The interfacial eigenvalue, although not shown in Figs 1 or 2, remains marginally stable at $\sigma = -\alpha i$ for all investigated values of the fluid parameters.

Figure 3 shows the effect of altering the extensional parameters ξ_j. The most obvious effect is that an increase in the value of ξ causes severe tilting of the eigenspectra towards the positive reals. This effect is more pronounced for the Oldroyd-B part but as ξ is increased then the distortion to the UCM part is such that the spectrum begins to turn back onto itself, the early stages of which are shown here in Fig. 3. Again the eigenspectrum for the two fluid problem, where only one fluid has a nonzero value for ξ, consists of one eigenspectrum overlying the other. It can be seen that the 'turning back' of the UCM spectrum is more pronounced for the two fluid problem and so the causes of this distortion to the spectrum are likely to be associated with the interactions between the two fluids.

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