# Approximate projection methods and time integration stability

C.D. Moen[a,*], S.P. Domino[b], G.J. Wagner[a]

[a] Sandia National Laboratories, Livermore, CA 94550, USA
[b] Sandia National Laboratories, Albuquerque, NM 87185, USA

**Abstract**

We examine a family of approximate projection methods for colocated, unstructured finite volume methods for the incompressible Navier–Stokes equations. Some combinations of pressure stabilization and projection time scale lead to unstable time integration. We explore the stability limits using numerical experiments.

*Keywords:* Approximate projection; Colocated, finite-volume; Incompressible Navier–Stokes equation

## 1. Introduction

Consider the class of approximate projection algorithms typically used to solve the incompressible Navier–Stokes equations on colocated, unstructured, finite-volume meshes. The momentum and continuity equations are split and solved in a segregated manner in order to simplify the solution procedure. The splittings can be abstracted to a general family of approximate factorizations. By design, the factorization errors are constructed to provide pressure stabilization (smoothing) and some means for computing a pressure that enforces mass balance. A time scale is required to relate velocity corrections in the mass balance to pressure corrections. This projection time scale is usually related to the time step or a characteristic time scale derived from the local velocity and mesh length scales.

Splitting approaches based on Helmholtz decomposition are typically derived from a semi-discrete form of the momentum equations and the projection scaling tends to be the time step [1,2,3,4]. Approaches based on matrix coefficients such as those in the SIMPLE family of schemes [5,6,7] use the fully discrete form of the momentum equations and the projection scaling tends to be a mix of the time step and a local characteristic time scale. We introduce the notion of a general projection time scale, $\tau$.

Most projection methods for colocated, finite-volume meshes are approximate. A projection method for a solenoidal velocity field is approximate when there are errors in the continuity equation and the discrete velocity field is not divergence-free. There can be additional splitting errors in the momentum equations. The error terms are consistent in that they reduce under mesh refinement. The terms not related to pressure stabilization can be reduced under nonlinear iteration. The splitting errors can be related to at least three behaviors [8]: the pressure field may have wiggles, there may be a time integration instability, and steady-state solutions may depend on the time step. We focus on the time integration stability issue here for time-marching procedures (primarily for steady problems).

A time step instability can occur when the projection time scale is smaller than the time step and is most apparent when the projection time scale is the convection scale. In our schemes, we select a variable time step such that we achieve a target CFL condition for the smallest convection time scale on the mesh. Convection is treated implicitly so we can take large time steps with $\mathrm{CFL} > 1$. The stability problem can be circumvented by using implicit under-relaxation, increasing the projection time scale, or adding stabilizing terms to the projection.

Several approximate projection methods are characterized in terms of time step stability. The methods are cast in an approximate factorization [9,10] form and we introduce a three-time-scale parameterization. We experiment with the limits of stability.

## 2. Numerical method

The numerical method is based on a finite-volume discretization of the equations on an unstructured finite-

* Corresponding author. Tel.: +1 925 294 3709; Fax: +1 925 294 3847; E-mail: cmoen@sandia.gov

© 2005 Elsevier Ltd. All rights reserved.
*Computational Fluid and Solid Mechanics 2005*
K.J. Bathe (Editor)

volume, vertex-centered mesh [11]. The results reported in this paper are generated on cell-centered meshes, but the projection formulations and the stability behaviors are similar to our vertex-centered results [8]. We use backward-Euler time integration with an implicit treatment for convected values and diffusion. Our control-volume approach is cast in conservative form and mass is balanced across each control volume.

The discrete momentum and continuity equations are written in matrix form in Eq. (1). The matrix **A** contains discrete, linearized contributions to the momentum equations from the time derivative, convection, and diffusion terms,

$$\begin{bmatrix} \mathbf{A} & \mathbf{G} \\ \mathbf{D} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{U}^{n+1} \\ \mathbf{P}^{n+1} \end{bmatrix} = \begin{bmatrix} \mathbf{f} \\ \mathbf{0} \end{bmatrix} \tag{1}$$

The discrete nodal gradient is **G** and the discrete nodal divergence is **D**. The function **f** contains the additional terms for the momentum equations. The velocity components are **U** and the pressure is **P**. The density is unity.

A general approximate factorization of Eq. (1) takes the form of

$$\begin{bmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{D} & \mathbf{B_1} \end{bmatrix} \begin{bmatrix} \mathbf{I} & \mathbf{B_2 G} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{A B_2 G} \\ \mathbf{D} & \mathbf{B_1} + \mathbf{D B_2 G} \end{bmatrix} \tag{2}$$

The factor $\mathbf{B_2}$ determines the projection time scale. We choose $\mathbf{B_2} = \tau$ to be a diagonal matrix, though the ideal form is $\mathbf{A}^{-1}$. The factor $\mathbf{B_1}$ defines the linear system for pressure. Ideally, $\mathbf{B_1}$ could be selected to cancel splitting errors in the continuity equation. Practically, the form of $\mathbf{B_1}$ is governed by implementation and linear solver efficiency.

We categorize the approximate projection methods as *unsmoothed, smoothed*, and *stabilized* depending on the relationship between $\mathbf{B_1}$, $\mathbf{B_2}$, and additional balancing terms. A *smoothed* scheme implies pressure stabilization and has good pressure smoothing properties. An *unsmoothed* scheme has no pressure stabilization and can admit pressure oscillations. A *stabilized* scheme is stable with respect to time integration for large time steps and usually has good pressure smoothing properties. We write a family of schemes using three projection time scale parameters $\tau_1$, $\tau_2$, and $\tau_3$.

The split momentum and continuity equations are

$$\begin{bmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{D} & -\mathbf{L}_1 \end{bmatrix} \begin{bmatrix} \mathbf{U}^{**} \\ \mathbf{P}^{**} \end{bmatrix} = \begin{bmatrix} \mathbf{f}(\mathbf{U}^n, \mathbf{U}^*) - \mathbf{G}\mathbf{P}^* \\ -\mathbf{L}_1\mathbf{P}^* + (\mathbf{L}_2 - \mathbf{D}\boldsymbol{\tau}_2\mathbf{G})\mathbf{P}^* \end{bmatrix} \tag{3}$$

where $\mathbf{L}_1 = \nabla\tau_1\nabla$ and $\mathbf{L}_2 = \nabla\tau_2\nabla$. The **L** operator is the discrete diffusion operator ($\mathbf{L} \neq \mathbf{DG}$). The superscripts * and ** imply guessed and intermediate states. The nodal correction is performed after the velocity and pressure solves,

$$\begin{bmatrix} \mathbf{I} & \tau_3\mathbf{G} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{U}^{n+1} \\ \mathbf{P}^{n+1} \end{bmatrix} = \begin{bmatrix} \mathbf{U}^{**} \\ \mathbf{P}^{**} \end{bmatrix} + \begin{bmatrix} \tau_3\mathbf{G}\mathbf{P}^* \\ \mathbf{0} \end{bmatrix} \tag{4}$$

The splitting errors relative to Eq. (1) are found by combining the matrix factors in Eqs. (3) and (4) (i.e., substitute the ** state):

$$\text{momentum} \quad \Rightarrow \quad [\mathbf{I} - \mathbf{A}\tau_3]\,\mathbf{G}\,(\mathbf{P}^{**} - \mathbf{P}^*) \tag{5}$$

$$\text{continuity} \quad \Rightarrow \quad [\mathbf{L}_1 - \mathbf{D}\tau_3\mathbf{G}]\,(\mathbf{P}^{**} - \mathbf{P}^*) + [\mathbf{L}_2 - \mathbf{D}\tau_2\mathbf{G}]\,\mathbf{P}^* \tag{6}$$

The pressure stabilization terms are scaled by $\tau_2$. The terms scaled by $\tau_1$ and $\tau_3$ are related to the pressure solve and velocity correction; the form of these two time scales seemingly affects time integration stability. In implementation, only two time scales are stored since cancellation requirements in Eqs. (5) and (6) lead to repeated values. The family of schemes is shown in Table 1.

Table 1
Each projection method in the family is defined by two time scale values and $\tau$ is either the time-step or a local characteristic time scale. Examples of related schemes are drawn from analogous staggered-mesh methods (SIMPLE), finite element methods, and other stabilization methods

| Category | Projection time scales | Related schemes |
|---|---|---|
| unsmoothed | $\tau_1 = \tau_3 = \Delta t$, $\tau_2 = 0$ | [1,2,3,4] |
| | $\tau_1 = \tau_3 = \tau_A$, $\tau_2 = 0$ | [6] |
| smoothed | $\tau_1 = \tau_2 = \tau_3 = \Delta t$ | [4] |
| | $\tau_1 = \tau_2 = \tau_3 = \tau_A$ | [5,7] |
| stabilized | $\tau_1 = \tau_A + \Delta t$, $\tau_2 = \tau_3 = \tau_A$ | [13] |
| | $\tau_1 = \tau_3 = \Delta t$, $\tau_2 = \min(\tau_A, \Delta t)$ | [14] |

## 3. Numerical experiments

The limits of projection time scales are explored for schemes susceptible to time integration instability, namely the *unsmoothed* and *smoothed* schemes (see Table 1). The velocity-driven cavity [12] is used as a demonstration case, though we also note similar behavior for other confined flows and open jets and plumes [11]. The mesh is a uniform 80×80 elements with Re = 100 based on lid velocity and cavity length. The solutions are time-marched from zero initial velocity to steady-state with an adaptive time step that maintains a constant value of the maximum nodal CFL number, $C$. The time step is $\Delta t = C{\cdot}\min(\tau_c)$, where $\tau_c$ is the convective time scale. We generally find that the time integration remains stable when the projection time scale is greater than or equal to the time step, so consider the measure $R = \Delta t/\min(\tau)$.

The first observation is that time step scaling always permits stable time integration for any CFL number. If the time scale is reduced, $\tau = \beta\Delta t$, then the method becomes unstable for $\beta < 1/2$, though this is not a universal result and is seen to be reduced in the Stokes regime. Generally, the limiting value is $R_{\text{lim}} = 2$. Results are confirmed for 2X steps up and down in mesh resolution and 10X steps up and down in Reynolds number.

A physical characteristic time scale often results when schemes are derived from matrix coefficients [6,7], $\tau_A = \tau_c\,(1 + 4/\text{Re}_\Delta)^{-1}$, where $\text{Re}_\Delta$ is a cell Reynolds number and we have approximated the convection and diffusion terms for a 2D problem. Applying a similar scale factor to characteristic time scaling, $\tau = \beta\tau_A$, the scheme can be made stable for $C > 1$, and $\beta \approx \frac{C}{R_{\text{lim}}}(1 + 4/\text{Re}_\Delta)$. The trend was confirmed at 10X steps up and down in Reynolds number, but does not hold in the Stokes regime.

Implicit relaxation, $\omega$, is often used to stabilize the use of the characteristic time scale [8]. The diagonal of the momentum matrix is scaled by $1/\omega$ and the projection time scale is scaled by $\omega$. The resulting ratio of artificial time step to relaxed projection time scale is

$$R \approx \frac{\omega(\Delta t^{-1} + (1-\omega)\max[(1+4/\text{Re}_\Delta)/\tau_c])^{-1}}{\omega \cdot \min[\tau_c(1+4/\text{Re}_\Delta)^{-1}]} \tag{7}$$

$$= \frac{C(1+4/\text{Re}_\Delta)}{1 + C(1-\omega)\,(1+4/\text{Re}_\Delta)} \tag{8}$$

The limiting relaxation behaves as $\omega \approx \min\left[1, 1 - \frac{1}{R_{\text{lim}}} + \frac{(1+4/\text{Re}_\Delta)^{-1}}{C}\right]$

Stability limitations disappear with a fully-coupled approach. A fully-coupled scheme can be derived from Eq. (3) by treating the pressure gradient in the momentum equations implicitly. Numerical experiments indicate that this family of schemes is stable for any selection of projection time scale and time step.

## 4. Concluding remarks

We define a general three-parameter formula for a family of approximate projection methods commonly found in literature. The schemes are different in their approaches to pressure stabilization and projection time scaling. We have performed numerical experiments to characterize stability boundaries for different projection time scales. The methods become unstable with respect to time integration when the projection time scale is smaller than the time step. The numerical experiments indicate some general limiting parameter relationships for stability. Under-relaxation is often used for stability and we relate the modified time scales to the limiting relationships. Future work will focus on a rigorous numerical analysis of the methods to prove the stability boundaries.

## Acknowledgements

Sandia is a multiprogram laboratory operated by Sandia Corporation, a Lockheed Martin Company, for the United States Department of Energy under Contract DE-AC04–94AL850000.

## References

[1] Chorin AJ. Numerical solution of the Navier–Stokes equations. Math of Comp 1968;22:745–762.
[2] Van Doormaal JP, Raithby GD. Enhancements of the SIMPLE method for predicting incompressible fluid flow. Num Heat Transfer 1984;7:147–163.
[3] Dwyer HA. Calculation of low mach number reacting flows. AIAA J 1990;28:98–105.
[4] Almgren AS, Bell JB, Crutchfield WY. Approximate projection methods: Part I. Inviscid analysis. SIAM J Sci Comp 2000;22:1139–1159.
[5] Rhie CM, Chow WL. Numerical study of the turbulent flow past an airfoil with trailing edge separation. AIAA J 1983;21:1525–1532.
[6] Patankar SV. Recent developments in computational heat transfer. J Heat Transfer 1988;110:1037–1045.
[7] Ferziger JH, Perić M. Computational Methods for Fluid Dynamics. New York: Springer-Verlag, 1996.
[8] Moen CD, Domino SP. A review of splitting errors for approximate projection methods. Paper AIAA 2003–4236, 16th Computational Fluid Dynamics Conference, Orlando, FL, June 2003.
[9] Dukowicz JK, Dvinsky AS. Approximate factorization as a high order splitting for the implicit incompressible flow equations. J Comp Phys 1992;102:336–347.
[10] Perot JB. An analysis of the fractional step method. J Comp Phys 1993;108:51–58.
[11] Moen CD, Evans GH, Domino SP, Burns SP. A multi-mechanics approach to computational heat transfer. Paper IMECE2002–33098, ASME International Mechanical Engineering Congress and Exposition, New Orleans, LA, November 2002.
[12] Ghia U, Ghia KN, Shinn CT. High-Re solutions for incompressible flow using the Navier–Stokes equations and a multigrid method. J Comp Phys 1982;48:387–411.
[13] Soto O, Löhner R, Cebral J. An implicit monolithic time accurate finite element scheme for incompressible flow problems. Paper AIAA 2001–2616, 15th AIAA Computational Fluid Dynamics Conference, Anaheim, CA, June 2001.
[14] Sahota, M. Personal communication, Los Alamos, NM: Los Alamos National Laboratory, 2004.