On the influence of diabatic effects on the motion of 3D-mesoscale vortices within a baroclinic shear flow

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Abstract

Three-dimensional concentrated vortices with vertical extensions throughout the whole troposphere and diameters corresponding to the sub-synoptic gradient wind regime are studied. In particular we focus on mechanisms that maintain the vertical coherent structure of such vortices within a baroclinic shear flow. Using matched asymptotic methods we derive equations for the horizontal velocity of the vertically oriented centerline of the vortices. We find that the vortices are not only advected by the background flow but that also their movement is influenced by diabatic effects. In this fashion, an initially vertically coherent vortex can be stabilized by diabatic effects even within a baroclinic shear flow.

Keywords: Concentrated vortices; Matched asymptotic methods; Baroclinic shear flow; Diabatic effects

1. Introduction

Eady [1] showed in his famous paper 'Long waves and cyclone waves' that within a stationary, zonal, and baroclinic background flow unstable waves travel with the mean unperturbed current. However, recent numerical and analytical studies have shown that the motion of geophysical vortices may be modified by various physical factors. For example, the secondary circulation (β- gyres) generated by vortices owing to the β- effect results in a self-induced translation relative to a uniform background flow. It has also been shown numerically by Raymond and Jiang [2] that factors such as latent heat affect the propagation of potential vorticity anomalies in such a way that they are able to propagate independently of the advecting wind. In this paper we study analytically the influence of diabatic effects on the motion of vortices. The derived results may contribute to a deeper understanding of cyclogenesis and the propagation of strong storms.

2. Governing equations

The starting points of derivation are the non-dimensional primitive equations consisting of mass continuity, thermodynamic equation, and momentum equations:

$$\frac{\partial \rho}{\partial t} + \vec{\nabla}_h \cdot (\rho\, \vec{v}_h) + \frac{\partial}{\partial z}(\rho\, w) = 0 \tag{1}$$

$$\left(\frac{\partial}{\partial t} + \vec{v}_h \cdot \vec{\nabla}_h + w\frac{\partial}{\partial z}\right)\Theta = S \tag{2}$$

$$Sr\frac{\partial}{\partial t}(\rho\vec{v}_h) + \vec{\nabla}_h \cdot (\rho\vec{v}_h\vec{v}_h) + \frac{\partial}{\partial z}(\rho\vec{v}_h w) + \frac{1}{M^2}\vec{\nabla}_h p + \frac{1}{Ro}(\vec{\Omega} \times \rho\vec{v})_h = 0 \tag{3}$$

$$Sr\frac{\partial}{\partial t}(\rho w) + \vec{\nabla}_h \cdot (\rho\vec{v}_h w) + \frac{\partial}{\partial z}(\rho w^2) + \frac{1}{M^2}\frac{\partial p}{\partial z} + \frac{1}{Ro}(\vec{\Omega} \times \rho\vec{v})_\perp + \frac{1}{Fr^2}\rho = 0 \tag{4}$$

The variables $\vec{v}_h$, ω, ρ, p, and Θ are functions of ($\vec{x}$, z, t) space and represent the horizontal and vertical velocities, density, pressure, and potential temperature, respectively. $\vec{\Omega}$ and γ are the vector of earth rotation and the ratio of specific heats. The variable S denotes a diabatic source term. The derivation of the non-dimensional equations is carried out using well-defined characteristic atmospheric values, independently of the characteristic length and time scales of any particular phenomena [3,4]. From considerations of a few universal non-dimensional parameters Klein [3] identifies a small parameter ϵ ranging between $\frac{1}{6}\cdots\frac{1}{8}$ in typical atmospheric flows. Relating ϵ to the dimensionless numbers Mach M, Rossby Ro, Froude Fr and Strouhal Sr, he

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proposes a distinguished limit $\epsilon \sim \frac{1}{Ro} \sim \sqrt{\mathrm{M}} \sim \sqrt{\mathrm{Fr}}$, $Sr \sim 1$ which serves as a basis for the asymptotic analysis.

In the subsequent derivations we study the movement of the vortices within a moving frame of reference with its origin located at the vortex center. Hence, with $\vec{x} = \epsilon^{-2}\vec{X} + \vec{\tilde{x}}$ we change into a $(\vec{\tilde{x}}, z, t)$ space where $\vec{X} = (X, Y)$ marks the position of the vortex center. The asymptotic expansion ansatz for the hydrodynamic variables $\mathcal{U} \in \{\vec{v}_h, w, \rho, \Theta\}$is chosen as follows:

$$\begin{aligned}\mathcal{U} &= \sum_i \epsilon^{(\frac{i}{2})}\mathcal{U}^{(\frac{i}{2})} \ \ (\epsilon^{\frac{3}{2}}\vec{\tilde{x}},z,\epsilon^2 t) \qquad i = 0,1,\ldots,n;\ n \in \mathcal{N} \\ &= \sum_i \epsilon^{(\frac{i}{2})}\mathcal{U}^{(\frac{i}{2})} \ \left(\epsilon^{\frac{3}{2}}\left[\vec{x} - \epsilon^{-2}\ \vec{X}(z,\epsilon^2 t)\right],z,\epsilon^2 t\right)\end{aligned} \tag{5}$$

With ansatz (5) the variables $\mathcal{U}$ are resolved on horizontal length scales of order $\mathcal{O}(100\,\mathrm{km})$ whereas the location of the vortex center is on synoptic scale, i.e. on length scales of order $\mathcal{O}(1000\,\mathrm{km})$. If we transform the coordinates from cartesian into cylindrical polar coordinates with $\tilde{x} = \epsilon^{-\frac{3}{2}}\, r \cos\theta$, $\tilde{y} = \epsilon^{-\frac{3}{2}}\, \mathrm{r}\ \sin\ \theta$, $\mathrm{z} = \mathrm{z}$ and $\epsilon^2 t = \tau$ then ansatz (5) takes the form:

$$\tilde{\mathcal{U}} = \sum \epsilon^{(i/2)}\tilde{\mathcal{U}}^{(i)}\ (r,\theta,z,\tau) \qquad i = 0,1,\ldots,n;\ n \in \mathcal{N} \tag{6}$$

From kinematic consideration [5] we obtain the asymptotic expansion for $\vec{X}$:

$$\vec{X} = \vec{X}^{(0)}(\tau) + \epsilon^{\frac{1}{2}}\vec{X}^{(\frac{1}{2})}(\tau,z) + \varepsilon^{\frac{2}{2}}\vec{X}^{(\frac{2}{2})}(\tau,z) + \ldots \tag{7}$$

Since we want to study concentrated vortices whose swirling velocity is faster than the background flow, we rescale the tangential velocity u_θ such that $\vec{v}_h = \vec{e}_r u_r + \vec{e}_\theta \epsilon^{-\frac{1}{2}} u_\theta$ and assume that $u_\theta^{(0)}$ is axisymmetric i.e. $u_\theta^{(0)} = u_\theta^{(0)}(r,z,\tau)$.

3. Results of asymptotic analysis

In this section we give a short review of the main results of our asymptotic analysis, which is strongly influenced by the work of Callegari and Ting [6]. In the orders $\mathcal{O}(\epsilon^{-\frac{i}{2}})$ with $i = 8, 7, 6, 5$ the vertical momentum equation yields hydrostatic balance, i.e. $\frac{\partial p^{(\frac{j}{2})}}{\partial z} = -\rho^{(\frac{j}{2})}$ with $j = 0, 1, 2, 3$. From the $\mathcal{O}(\epsilon^{\frac{1}{2}})$ horizontal momentum equation one gets the well known gradient wind relation

$$\frac{1}{\rho^{(0)}}\frac{\partial p^{(\frac{6}{2})}}{\partial r} - \frac{u_\theta^{(0)^2}}{r} - \Omega_0 u_\theta^{(0)} = 0. \tag{8}$$

From mass continuity we obtain in the orders $\mathcal{O}(\epsilon^{\frac{j}{2}})$ with $j = 0, 1, 2$ zero vertical velocity, i.e. $\mathrm{w}^{(\frac{j}{2})} = 0$ with $j = 0, 1, 2$. However, further analysis of the thermodynamic equation yields in the order $\mathcal{O}(\epsilon^{\frac{3}{2}})$

$$w^{(\frac{3}{2})} = S^{(\frac{3}{2})} / \frac{\partial\theta^{(\frac{3}{2})}}{\partial z} \tag{9}$$

Hence, the vertical velocity is induced by a diabatic source term $S^{(\frac{3}{2})}$ and its magnitude depends on stratification $\frac{\partial\theta^{(\frac{3}{2})}}{\partial z}$. This is a well-known relation in the context of the 'weak temperature gradient approximation', see e.g. [4,7]. In the subsequent analysis, we assume $S^{(\frac{3}{2})} = S^{(\frac{3}{2})}(r,z,\tau)$ to be axisymmetric and thus $w^{(\frac{3}{2})} = w^{(\frac{3}{2})}(r,z,\tau)$.

From the $\mathcal{O}(\epsilon^{\frac{3}{2}})$ mass continuity we obtain divergence flow conditions

$$\rho^{(0)}\vec{\nabla}_h^* + \frac{\partial(\rho^{(0)}w^{(\frac{3}{2})})}{\partial z} = -\rho^{(0)}\left(X_z^{(\frac{1}{2})}\cos\theta + Y_z^{(\frac{1}{2})}\sin\theta\right)\ \frac{\partial w^{(\frac{3}{2})}}{\partial r} \tag{10}$$

where $\vec{\nabla}_\mathrm{h}^* = (\frac{\partial u_r^{(0)}}{\partial r} + \frac{u_r^{(0)}}{r} + \frac{1}{r}\frac{\partial u_\theta^{(\frac{1}{2})}}{\partial\theta})$. Using the fact that a divergent flow can be decomposed into a divergence-free $(u_r^{(0)}, u_\theta^{(\frac{1}{2})})^{nd}$ and an irrotational $(u_r^{(0)}, u_\theta^{(\frac{1}{2})})^{d}$ part with $u_\theta^{(\frac{1}{2})^{nd}} = -\frac{\partial\psi^{(\frac{1}{2})}}{\partial r}$ and $u_r^{(0)^{nd}} = \frac{1}{r}\frac{\partial\psi^{(\frac{1}{2})}}{\partial\theta}$ we derive from the $\mathcal{O}(\epsilon^{\frac{2}{2}})$ horizontal momentum equations, equations for the Fourier coefficients of $\psi^{(\frac{1}{2})}$

$$\begin{aligned}&u_\theta^{(0)}\left(\frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial r^2} - [\frac{\zeta_r^{(0)}}{u_\theta^{(0)}} + \frac{j^2}{r^2}]\right)\ \psi_{jk}^{(\frac{1}{2})} = \\ &\delta_{j1}T_k\left[w_0^{(\frac{3}{2})}(\zeta^{(0)} + r\frac{\partial^2 u_\theta^{(0)}}{\partial r^2}) - r\Omega_0\frac{\partial w_0^{(\frac{3}{2})}}{\partial r}\right] - \delta_{j1}r\zeta_r^{(0)}u_{\theta,jk}^{d(\frac{1}{2})}\end{aligned} \tag{11}$$

where $j = 1,2,\ldots n$; $k = 1,2$; $T_1 = \frac{\partial X^{(\frac{1}{2})}}{\partial z}$, $T_2 = \frac{\partial Y^{(\frac{1}{2})}}{\partial z}$, δ_{j1} is the Kronecker delta, $\zeta^{(0)} = \frac{1}{r}\frac{\partial}{\partial r}(ru_\theta^{(0)})$ denotes the leading order vorticity and $\zeta_r^{(0)} = \frac{\partial\zeta^{(0)}}{\partial r}$

The boundary conditions for solving Eq. (11) is obtained by considering the $\mathcal{O}(\epsilon^{\frac{2}{2}})$ horizontal momentum equations in the limit $r \to 0$ without assuming 'solid body rotation' condition. Hence, for $j = 1$ the solution of Eq. (11) is

$$\begin{aligned}\psi_{1k}^{(\frac{1}{2})} \ = \ & 2u_\theta^{(0)}\left[\frac{\frac{\partial\psi_{1k}^{(\frac{1}{2})}}{\partial r}}{\zeta^{(0)}}\right]_{r=0} - u_\theta^{(0)}T_k\int_0^{\bar{\bar{r}}}\frac{r}{u_\theta^{(0)2}}\Omega_0 w_0^{(\frac{3}{2})}dr + \\ & u_\theta^{(0)}T_k\int_0^{\bar{\bar{r}}}\frac{1}{\bar{r}u_\theta^{(0)2}}\left[2\int_0^{\bar{r}} w_0^{(\frac{3}{2})}(u_\theta^{(0)} + \Omega_0 r)dr\right]\ d\bar{r}.\end{aligned} \tag{12}$$

Equations for the movement of the vortex center are obtained using matched asymptotics. For this purpose we have considered Eq. (12) as an inner solution. The far field behavior of the leading order flow is obtained by solving the conservation of quasigeostrophic potential vorticity $q = q\ (\hat{r}, z, \tau)$, i.e. $\frac{dq}{d\tau} = 0$ where

$$q = \frac{1}{\hat{r}}\frac{\partial}{\partial \hat{r}}\left(\hat{r}\frac{\partial \hat{\psi}}{\partial \hat{r}}\right) + \beta \hat{r}\sin\theta + \frac{\Omega_0^2\Theta_\infty}{\Theta_z^{(2)}}\frac{\partial^2 \hat{\psi}}{\partial z^2} - \frac{\Omega_0^2\Theta_\infty}{\rho^{(0)}}\frac{\partial}{\partial z}\left(\frac{\rho^{(0)}}{\Theta_z^{(2)}}\right)\frac{\partial \hat{\psi}}{\partial z} \tag{13}$$

and $\hat{\psi}$ is the stream function describing the geostrophic wind. The conservation of potential vorticity can be derived from Eqs. (1)–(4) using the asymptotic ansatz

$$\mathcal{U}_{QG} = \sum \epsilon^i \mathcal{U}_{QG}^{(i)}(\epsilon^2\vec{x}, z, \epsilon^2 t)$$

Seeking a singular vortex solution $\hat{\psi}_s$ on the f -plane, i.e. $\beta = 0$, embedded in a continuous background flow Ψ we find that the outer stream function $\hat{\psi}$ can be written as

$$\hat{\psi} = \hat{\psi}_s + \Psi = \frac{\Gamma}{2\pi}\ln\hat{r} + \Psi \tag{14}$$

The variable $\Gamma = \Gamma(z)$ denotes a vertically varying circulation of the vortex.

Finally, from the matching condition $\lim_{r\to\infty}\psi = \lim_{\hat{r}\to 0}\hat{\psi}$ we obtain equations for the movement of the vortices

$$U = \frac{\partial X}{\partial \tau} = +\frac{\partial \Psi}{\partial y} - \frac{\partial \vec{X}^{(\frac{1}{2})}}{\partial z}\lim_{r\to\infty}\left(\int_0^r w_0^{(\frac{3}{2})}(u_\theta^{(0)} + \Omega_0 r)\,dr\right) \tag{15}$$

$$V = \frac{\partial Y}{\partial \tau} = -\frac{\partial \Psi}{\partial x} - \frac{\partial \vec{Y}^{(\frac{1}{2})}}{\partial z}\lim_{r\to\infty}\left(\int_0^r w_0^{(\frac{3}{2})}(u_\theta^{(0)} + \Omega_0 r)\,dr\right)$$

Thus we have shown that the vortex is not being passively advected by the local steering level winds, but additionally the movement is due to $w_0^{(\frac{3}{2})}$ modified by diabatic effects. Note, in order to carry out the matching we had to ensure the convergence of the integrals in Eq. (12) for large r. It can be shown that in the case of an algebraic decay for $w_0^{(\frac{3}{2})}$ (e.g. $w_0^{(\frac{3}{2})} \le o(\frac{1}{r^\alpha})$, $\alpha \ge 3$ as $r \to \infty$) and with a sufficiently rapid decay for the leading order vorticity $\zeta^{(0)}$ (e.g. $\zeta^{(0)} \le \mathcal{O}(\exp(-r^2))$, as $r \to \infty$) the inner solution (Eq. (12)) is convergent in the limit $\bar{\bar{r}} \to \infty$ and takes the form

$$\psi_{1k}^{(\frac{1}{2})} = \mathcal{O}\left(\frac{1}{\bar{\bar{r}}}\right) + T_k\bar{\bar{r}}\int_0^{\bar{\bar{r}}} w_0^{(\frac{3}{2})}(u_\theta^{(0)} + \Omega_0 r)dr \qquad \text{as} \quad \bar{\bar{r}} \to \infty \tag{16}$$

Further analysis of higher orders in the asymptotics will yield core structure evolution equations in analogy with the work by [6]. We will have to verify that the rapid radial decay assumption on the leading order vorticity is consistent with these equations.

4. Concluding remarks

We have shown that the movement of vortices is influenced by diabatic effects to which latent heat release is a major contributor. Updrafts of moist air and subsequent condensation due to the adiabatic cooling effect may provide the means to stabilize strong vertically coherent vortices even in the presence of vertical shear.

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