A FEM Navier–Stokes solver coupled to a front-tracking algorithm for two-phase flows

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Abstract

A front-tracking algorithm for interfacial flows is reviewed. The interface is described by a set of closed lines. Fixed markers are set at the crossing points of these lines, while auxiliary markers are defined at the intersection points with the grid cell faces and inside each computational cell. These auxiliary markers are added and removed dynamically by a local area conservation algorithm. A Navier–Stokes solver that includes this front-tracking algorithm has been developed for 2D incompressible two-phase flows based on the finite element method. A variational formulation of the surface tension term, which removes the singularity of the capillary force, is proposed. Spurious currents are greatly reduced and oscillating drop dynamics is accurately reproduced.

Keywords: Interface tracking; Two-phase flow; Finite element Navier–Stokes solver, Surface tension; Parasitic currents

1. Introduction

Multiphase and free-surface flows play an important role in many different natural and industrial processes, such as, for example, combustion, boiling and condensation, sprays, and direct-injection engines. Several numerical methods have been devised and used to model complex 2D and 3D flows exhibiting topology changes. In this paper we report on a new approach by Aulisa et al. [1,2] that represents an interface line in 2D as a series of segments connecting an ordered list of interface markers. The method combines a Lagrangian front-tracking technique with a local redistribution algorithm that conserves the area spanned by the interface line and spreads the markers uniformly along the interface, eventually by adding and removing them locally when this is required by the interface evolution. We have also generalized this algorithm in a rather natural way to unstructured quadrangular grids, with negligible reduction in performance [3]. In 3D, the interface lines define a coarse deforming Lagrangian quadrangular mesh. The lines are advected by the flow and are reconstructed separately. We have also developed a FEM solver of the incompressible Navier–Stokes equations that includes the front-tracking algorithm. Particular attention has been given to an accurate and mathematically sound description of the surface tension term and the associated pressure jump. Surface tension forces are localized on the interface between two fluids and a stable numerical algorithm is even more difficult in the presence of high density and viscosity ratios. A weak formulation of the capillary forces has produced good results in both stationary problems, with a very low level of spurious currents, and axisymmetric droplet oscillations [4].

2. Equation for interface advection

Let Ω be a bounded domain with the reference phase contained in the subdomain $\Omega_1 \subset \Omega \subset \Re^3$ and χ_1 the indicator function for the reference phase

$$\mathcal{X}_1(t,\vec{x}) = \int_{\Omega_1(t)} \delta(\vec{x}-\vec{x}')\, d\vec{x}' \tag{1}$$

The integral is over the volume $\Omega_1(t)$ bounded by the interface $\Gamma_1(\vec{x}, t)$. The indicator χ_1 is one inside Ω_1, zero outside, discontinuous across the interface and satisfies

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Computational Fluid and Solid Mechanics 2005
K.J. Bathe (Editor)

the advection equation $D_\chi 1(\vec{x}, t)/Dt = 0$. The solution of the advection equation can be found as a function of the initial condition χ_{10} by using the method of the characteristics

$$\mathcal{X}_1(\vec{x},t) = \mathcal{X}_{10}(\vec{x}_0,t_0); \quad \vec{x}(t) = \vec{x}_0 + \int_{t_0}^{t} \vec{u}(\vec{x}(t'),t')dt' \tag{2}$$

Then, given the initial position of the interface, we can follow its motion by simply integrating Eq. (2).

3. Marker advection and redistribution algorithm

The interface Γ_1 is described by a coarse Lagrangian surface mesh with quadrangular elements. The nodes of these elements are gathered together to define a set of closed lines. Each line consists of an ordered list of points connected by segments. Each list includes some fixed markers which are located on the nodes of the quadrangular elements where different lines cross each other. Intersection markers are added where the segments cross the cell faces of the computational fixed grid, while conservation markers are added inside each computational cell to conserve area and volume. Each line is advected and reconstructed separately, by moving all its markers to their new positions. New intersection markers are then computed. The procedure of removing and adding internal points is a direct extension to the three-dimensional space of the 2D technique described in Aulisa et al. [1]. The result of this procedure is to substitute the old intersection and area conservation markers with new area conservation markers which conserve the area in the direction perpendicular to the line. More details can be found in Aulisa et al. [2].

4. Numerical tests

We have extensively tested our method with standard not deforming velocity fields, such as translations and solid body rotations, and not uniform vorticity fields as well. In particular, in Cartesian grids the errors we obtain for the 2D single vortex test) described in Rider et al. [5], are between two and three orders of magnitude lower than those reported for standard VOF and level set methods [1,2]. We have also extended this method to unstructured quadrangular grids, and for the same test the deterioration in the performance due to the irregular shape of the cells is rather modest and contained within a factor of 2.5. More results on this extension can be found in Aubert et al. [3]. As an example of the performance of our technique in 3D, we consider an incompressible flow field that is the superposition of three two-dimensional single vortex fields, each of them deforming and stretching a fluid body on a different coordinate plane, see Aulisa et al. [2]. We position a cylinder in the above field and advect it with a cosinusoidal time modulation with period $T = 2$, so that at $t = 2$ the fluid body should be back to its original position. In Table 1 we present the volume and geometric errors as a function of both the CFL number and the grid resolution. In the considered ranges of time steps and grid spacing the errors are rather good.

Table 1
Rotation cylinder test

n	CFL	E_v	E_g
32	1.00	4.17e–3	8.76e–4
32	0.10	6.54e–4	1.53e–5
32	0.01	7.78e–5	5.60e–6

n	CFL	E_v	E_g
16	0.10	1.98e–3	9.94e–5
32	0.10	6.54e–4	1.53e–5
64	0.10	6.25e–5	2.45e–6

5. Dynamics with Navier–Stokes equations

Let Ω be a 2D domain with Lipswich-continuous boundary Γ. The reference phase domain Ω_1 consists of a series of simply connected open domains with boundaries in C^2 whose union is denoted by Γ_1. Let ρ_1, μ_1, ρ_2 and μ_2 be the constant density and viscosity for the reference and secondary phases, respectively. Therefore, the density ρ and the viscosity μ in Ω can be defined by $\rho = \rho_1\chi_1 + (1 - \chi_1)\rho_2$ and $\mu = \mu_1\chi_1 + (1 - \chi_1)\mu_2$. Finally, let $\vec{u}$ be the velocity field and p the pressure. If we multiply the incompressible Navier–Stokes equations by the respective test functions $\vec{v} \in \mathbf{H}^1(\Omega), r \in L_0^2(\Omega)$ and integrate by parts, we get

$$\int_\Omega \rho \frac{\partial \vec{u}}{\partial t} \cdot \vec{v}\, d\vec{x} + \int_\Omega \mu \vec{\nabla}\vec{u} : \vec{\nabla}\vec{v}\, d\vec{x} + \int_\Omega \vec{v}\rho(\vec{u}\cdot\vec{\nabla})\vec{u}\, d\vec{x} - \int_\Omega \vec{\nabla}\cdot\vec{v}p\, d\vec{x} + \int_{\Omega_1} \sigma\vec{v}\cdot\vec{\nabla}\hat{k}\, d\vec{x} + \int_{\Omega_1} \sigma\hat{k}\vec{\nabla}\cdot\vec{v}\, d\vec{x} = \int_\Omega \vec{f}\cdot\vec{v}\, d\vec{x} \quad \forall \vec{v} \in \mathbf{H}^1(\Omega) \tag{3}$$

$$\int_\Omega \vec{u} r\, d\vec{x} = 0 \quad \forall, r \in L_0^2(\Omega) \tag{4}$$

where σ is the surface tension and $\vec{f}$ the body force. The function $\hat{k}$ is the extension to Ω_1 of the curvature κ, which is defined only on the interface Γ_1. The system (3–4) must be solved for $(\vec{u}, p)$ on Ω with given coefficients ρ, μ, $\hat{k}$. By tracking the evolution of Ω_1 and its surface Γ_1, as discussed in the previous sections, we can compute the indicator function χ_1 $(\vec{x}, t)$ and then the density ρ and the viscosity μ. The extended curvature function $\hat{k} \in H^1(\Omega)$ may be any function in Ω_1 such as $\hat{k} = \kappa, \forall\, \vec{x} \in \Gamma_1$. The interface line is described by the equation $\vec{x} = \vec{x}(s)$ where the parameter s is the arc length, then the tangent $\vec{t}$, the normal $\vec{n}$ and the curvature κ are related by $\vec{t} = d\vec{x}/ds$ and $\vec{\eta} = \kappa\,\vec{n} = d^2\vec{x}/ds^2$. Since only the first derivative of $\vec{x}(s)$ is numerically easily available it is better to solve $\vec{\eta} = k\,\vec{n}$ from its weak form

$$\int_{\Gamma_1} \vec{\eta} \cdot \vec{v}\, ds + \int_{\Gamma_1} \frac{d\vec{\eta}}{ds} \cdot \frac{d\vec{v}}{ds} ds = [\vec{t} \cdot \vec{v}]_{\partial\Gamma_1} - \int_{\Gamma_1} \frac{d\vec{\eta}}{ds} \cdot \frac{d\vec{v}}{ds} ds \quad \forall \vec{v} \in \mathbf{H}^1(\Gamma_1) \tag{5}$$

and then compute $\kappa = |\vec{\eta}|$. The parameter $\beta > 0$ implies $\kappa \in H^1$ (Γ) and it can be used as a regularization parameter when the r.h.s. function $d\vec{x}/ds$ is not regular enough. As $\beta \to 0$, the exact curvature κ can be recovered.

6. Steady-state solutions and spurious current tests

Dynamical situations close to equilibrium are useful to detect slowly non-convergent numerical schemes for surface tension computations which are usually masked and easily overlooked by flows. All discretizations of the surface tension term show spurious currents which are unphysical flows generated near the interface. The magnitude of these currents can be estimated by the simulation of a static spherical drop/bubble in zero gravity [6]. The analytical solution has a zero velocity field and a pressure jump across the interface due to surface tension whose value is given by the Laplace's law $\Delta P = \sigma\,\kappa$. In the weak representation of the Navier–Stokes equations, Eqs. (3–5), the singular surface tension term on Γ_1 has been transformed into a regular volume term over Ω_1 and the spurious currents and the pressure fluctuations are greatly reduced.

7. Axisymmetrical drop oscillations test

We consider axisymmetrical oscillations of a viscous drop subject to a small perturbation of the second spherical harmonics in no-gravity condition. Our investigation is primarily focused on the period and damping coefficient of the oscillations. Some preliminary computations have been made in order to check the ability of our numerical approach to accurately solve the problem. A comparison with the theoretical predictions by Prosperetti [7] and Lamb [8] is presented in Table 2. The results, obtained with 64 × 64 mesh resolution, are in good agreement for all density ratios expecially at low values where they compare favorably with those obtained with other surface tension representations. The results are rather sensitive to the curvature computation and the value 0.001 for the parameter β appears to be a good compromise. In the same table we also compare the period T of the linear analytical theory with the computed one for the density ratio $\rho_1/\rho_2 = 0.01$. The convergence rate with mesh resolution is very good.

8. Conclusion

Table 2
Period for different density ratios at $\beta = 0.001$ 64 × 64 (on the left) and for different resolutions at $\beta = 0.001$ and $\rho_1/\rho_2 = 0.01$ (on the right)

ρ_1/ρ_2	linear T	$\Delta t = .025$ T	err %
1	4.05	4.30	6.2
.1	3.24	3.38	4.3
.01	3.15	3.26	3.5
.001	3.14	3.22	2.5
.0001	3.14	3.21	2.2

grid	linear T	computed T	err %
64^2	3.152	3.261	3.5
128^2	3.152	3.222	2.2
256^2	3.152	3.165	0.4

In this paper we have briefly reviewed a front-tracking method for interface advection in the two-dimensional and three-dimensional space which is able to follow a fluid body deforming and stretching in a complex vortical flow with great accuracy. We have also briefly described a new solver for the Navier–Stokes equation based on the finite element method. Preliminary results show that spurious currents are greatly reduced and that bubble oscillations can be reproduced in a very accurate way.

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