Solenoidal invariance of the dynamics for stability calculations

J.I.H. López[a,*], J.R. Meneghini[b], J.A.P. Aranha[b], F. Saltara[b]

[a]*Department of Mechanical Engineering, Mackenzie Presbyterian University, Rua da Consolacao, 930, Consolacao, São Paulo, 01302–907, Brazil*
[b]*NDF, Department of Mechanical Engineering, University of São Paulo-USP, Butanta, 05508–900, Brazil*

Abstract

The purpose of this study is to investigate the influence of the size of a discrete solenoidal subspace, generated by the kernel of the divergence operator, in the dynamic behavior of a stability problem. This problem is associated with the incompressible and viscous flow around a circular cylinder. The solenoidal subspaces were generated from the quadratic Taylor-Hood element for the velocity and the linear element for the pressure. These discrete subspaces were characterized according to their dimensions and their ability to catch the dynamics of the problem of linear stability.

Keywords: Frechet operator; Hydrodynamic stability; Solenoidal subspace; Navier–Stokes; Finite element method; circular cylinder spectra

1. Introduction

In this work we investigate the dynamics of the Frechet operator in the field of hydrodynamic stability for the different types of discretization that are common in the finite element formulation.

As is known, the law of mass conservation for incompressible fluids reveals that when the operator is discretized the solution must be found in a solenoidal subspace [1]. Previous works have reported some solutions for the Navier–Stokes equations in such subspace [2,3]. However, these solutions are strongly dependent on the size of the solenoidal subspace [2,4].

An important question is how to solve the persistence of the pair eigenvalue-eigenvector in the stability calculations when the dimension of a solenoidal subspace is changing. These changes are usually due to the type of discretization. In particular, in the case of the incompressible fluid flowing around a circular cylinder, we show that the stability problem is conserved when the dimension of the solenoidal subspace is highly reduced.

The theorem of espectral decomposition reads: Let A be a linear self-adjoint differential operator:

$$A\,u(x) \;=\; f(x), \quad a \;\leq\; x \;\leq\; b \tag{1}$$

where the function $f(x)$ is known. The solution of Eq. (1) can be written in the form

$$u(x) \;=\; A^{-1}f(x) \;=\; \sum_{i=1}^{\infty} \frac{(f(x),u_i)}{\lambda_i} u(x)_i \tag{2}$$

with $\lambda_1 \leq \lambda_2 \leq \ldots \lambda_m \leq \ldots$ being a countable infinite number of real eigenvalues and $u_1, u_2, \ldots, u_m, \ldots$ being a complete orthonormal set of eigen-functions.
This result reveals that the pair:

$$(\lambda_i, u(x)_i) \tag{3}$$

contains the information of the system dynamics [5,6,7]. Although the formal solution is obtained in the pairs of Eq. (2), the calculation is very difficult, so other methodologies are prefered.

The present work is organized as follows: in the next section we define the stability problem. In Section 3 we review the weak formulation of the linear stability problem for Navier–Stokes equations. In Section 4 we study the projection mechanism in the solenoidal subspace. Finally, we discuss the numerical solutions and results.

2. The problem

We calculated the steady flow around the circular cylinder, $\mathbf{U}(\mathrm{x},Re)$ and the scalar field $p = (\mathbf{x}, Re)$ [8,9,10,11,12]. For the stability calculation it is

* Corresponding author. Tel.: +55 (11) 3236 8756; E-mail: jihlopez@mackenzie.com.br

Computational Fluid and Solid Mechanics 2005
K.J. Bathe (Editor)

important to study the evolution of a disturbance **u** of **U**, which is governed [7,13] by

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{U}\cdot\nabla)\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{U} + (\mathbf{u}\cdot\nabla)\mathbf{u} = -\nabla p + \frac{1}{Re}\nabla^2\mathbf{u} \tag{4}$$

$$\nabla\cdot\mathbf{u} = \mathbf{0} \tag{5}$$

with prescribed initial condition $\mathbf{u(x,0)} = \mathbf{u_0}$. Equations (4) and (5) can be written as

$$\frac{\partial \mathbf{u}}{\partial t} = \mathbf{F}(\mathbf{u}, Re) - \nabla p \tag{6}$$

$$\nabla\cdot\mathbf{u} = \mathbf{0} \tag{7}$$

where

$$\mathbf{F}(\mathbf{u}, Re) = \mathbf{F_u}(0, Re|\mathbf{u}) - \mathbf{u}\cdot\nabla\mathbf{u} \tag{8}$$

and

$$\mathbf{F_u}(0, Re|\mathbf{u}) = \frac{1}{Re}\nabla^2\mathbf{u} - \mathbf{U}(Re)\cdot\nabla\mathbf{u} - \mathbf{u}\cdot\nabla\,\mathbf{U}(Re) \tag{9}$$

Equation (9) is a linear operator (Frechet). When **u** is sufficiently small, we seek for conditions for the stability of **U** from the linearized problem

$$\frac{\partial \mathbf{u}}{\partial t} = \mathbf{F_u}(0, Re|\mathbf{u}) \tag{10}$$

which arises from Eqs. (6–9) when $\mathbf{u}\cdot\nabla\mathbf{u}$ is set to zero. Expressing $\mathbf{u} = e^{\sigma t}\zeta$, and $p[\mathbf{u}] = e^{\sigma t}p[\zeta]$, and inserting in Eq. (10), it results that ζ and $\sigma(Re) = \xi(Re) + i\eta(Re)$ satisfy the spectral problem

$$\sigma\zeta = \mathbf{F_u}(0, Re|\zeta) - \nabla p[\zeta] \tag{11}$$

When Ω is a bounded domain there is an infinite number of isolated $\sigma(Re)$ and all lie inside a parabola opening out to the left in the complex $\sigma(Re)$ plane. It is known, from the linearization principle, that the proof of the stability or instability in a basic flow with respect to small perturbations is reduced to the complete determination of the spectrum of the linear problem represented by Eq. (11) [14,15]. It is well known that, in the solenoidal subspace,

$$\mathbf{u}\in\mathbf{H} = \{\mathbf{u} : \nabla\cdot\mathbf{u} = 0, \mathbf{u}\Big|_{\partial\Omega} = 0, \left\langle|\nabla\mathbf{u}|^2\right\rangle < \infty\} \tag{12}$$

therefore Eq. (11) becomes [13]:

$$\sigma\zeta = \mathbf{F_u}(0, Re|\zeta) \tag{13}$$

3. Weak formulation

A weak formulation of Eq. (11) results in the following equations [16]:

$$\sigma(\mathbf{u},\mathbf{v}) + a(\mathbf{u},\mathbf{v}) + b(\mathbf{v},p) + c(\mathbf{U},\mathbf{u},\mathbf{v}) + c(\mathbf{u},\mathbf{U},\mathbf{v}) = 0 \tag{14}$$

$$b(\mathbf{u},q) = 0 \tag{15}$$

where the forms are defined by

$$a(\mathbf{u},\mathbf{v}) = \int_\Omega \nabla\mathbf{u} : \nabla\mathbf{v}\, dx \tag{16}$$

$$b(\mathbf{v},q) = -\int_\Omega (\nabla\cdot\mathbf{v})q\, dx \tag{17}$$

$$c(\mathbf{u},\mathbf{v},\mathbf{w}) = \int_\Omega (\mathbf{u}\cdot\nabla)\mathbf{v}\cdot\mathbf{w}\, dx \tag{18}$$

The functions **v** and q belong to the spaces Υ and Λ, where

$$\Upsilon = \{\mathbf{v}\in\mathbf{W}_2^1(\Omega) : \mathbf{v}|_{\partial\Omega} = 0\} \tag{19}$$

and

$$\Lambda = \{q\in L^2(\Omega) : \int_\Omega q(x)\, dx = 0\} \tag{20}$$

Here L^2 and $\mathbf{W}_2^1$ denote, correspondingly, the usual Sobolev spaces of the square-integrable functions and the functions whose first derivatives are square-integrable. To discretize Eq. (11), we first partition Ω with the finite element grid Γ_h, which consists of triangles. On this grid the spaces are defined as

$$\mathbf{X}_h^2 = \{w_h\in\mathbf{C}^0(\Omega)|w_h|_\kappa \in \mathbf{P}_2(K), \forall K\in\Gamma_h\} \tag{21}$$

Now, two conforming approximation of $\Upsilon\times\Lambda$, namely $\Upsilon_\mathrm{h}\times\Lambda_\mathrm{h}$, can be built, where $\Upsilon_\mathrm{h} = (\mathbf{X}_h^2)^2\cap\Upsilon$ and $\Lambda_h = \mathbf{X}_h^1$. This corresponds to the popular Taylor-Hood formulation (P2P1) [17,18]. Using finite elements to solve Eqs. (14) and (15) results in the generalized eigenvalue problem. Let us consider $\mathbf{w}\in\mathbb{R}^n$ as the vector of nodal degrees of freedom defining the velocity perturbation **u**, and $\mathbf{q}\in\mathbb{R}^m$ the vector of degrees of freedoms defining the pressure perturbation p. The discretized form of Eqs. (14 and 15) is

$$\begin{pmatrix}\mathbf{A}+\mathbf{L}(\mathbf{w})-\mathbf{R}\\ \mathbf{R}^T \quad\quad 0\end{pmatrix}\begin{pmatrix}\mathbf{w}\\ \mathbf{q}\end{pmatrix} = \sigma\begin{pmatrix}\mathbf{M} & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}\mathbf{w}\\ \mathbf{q}\end{pmatrix} \tag{22}$$

where **A** is a $(n\times n)$ sparse viscosity matrix, $\mathbf{L}(\mathbf{w})$ a nonsymmetric $(n\times n)$ matrix, the sum of the convection terms, $-\mathbf{R}$ is the discrete gradient $(n\times m)$ matrix of rank m, $\mathbf{R}^T$ is the discrete divergence $(m\times n)$ matrix operator,

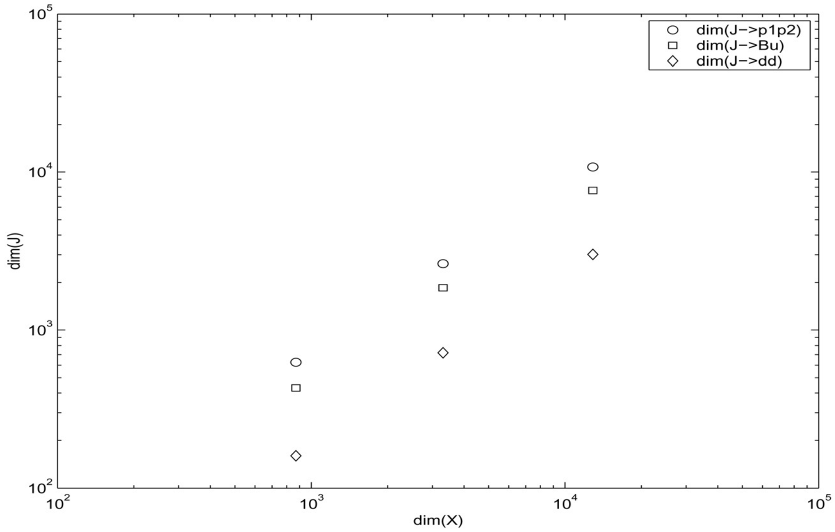

Fig. 1. Dimensions of solenoidal subspaces.

and $\mathbf{M}$ is a mass ($n \times n$) symmetric matrix. The eigenvalue problem, Eq. (22), is real and non-symmetric, so eigenvalues will be either real or they can occur in complex conjugate pairs [9,17]. The spectrum of the matrix $\mathbf{A} + \mathbf{L}(\mathbf{w})$ are the eigenvalues and eigenvectors of the restriction of this operator $\mathbf{A} + \mathbf{L}(\mathbf{w})$ in the subspace of Υ_{h}, defined by

$$\ker \mathbf{R}^T = \{\mathbf{w}_h \in \Upsilon_h | \int_\Omega \nabla \cdot \mathbf{w}_h q_h dx = 0, \forall q_h \in \mathbf{\Lambda}_h\} \quad (23)$$

consisting of vectors whose discrete divergence is zero.

4. Decoupling procedure

The decoupling can be achieved by finding a basis $\{\mathbf{t}_1, \ldots, \mathbf{t}_{\dot{\mathrm{n}}}\}$ of ker $\mathbf{R}^T$, where $\dot{\mathrm{n}} = \dim(\mathbf{R}^T)$. It is known that $\mathbf{R}^T$ has a full rank, $\dot{\mathrm{n}} = n - m$ [19]. Having such basis, $\mathbf{u}$ can then be written as

$$\mathbf{u} = \sum_{i=1}^{\dot{\mathbf{n}}} \dot{\mathbf{u}}_\mathbf{i}\, \mathbf{t}_\mathbf{i} = \mathbf{T}^T \dot{\mathbf{u}} \quad (24)$$

for some $\dot{\mathbf{u}} \in \mathbb{R}^{\dot{\mathrm{n}}}$, where $\mathbf{T}$ denotes the ($\dot{n} \times n$) matrix with rows $\mathbf{t}_1^T, \ldots, \mathbf{t}_{\dot{n}}^T$. Since

$$\mathbf{T} \cdot \mathbf{R} = (\mathbf{R}^\mathbf{T} \cdot \mathbf{T}^\mathbf{T})^\mathbf{T} = \mathbf{0} \quad (25)$$

and multiplying the first row of Eq. (22) by $\mathbf{T}$, it can be shown that

$$\dot{\mathbf{A}}\, \dot{\mathbf{u}} = \sigma\, \dot{\mathbf{M}}\, \dot{\mathbf{u}} \quad (26)$$

where $\dot{\mathbf{A}} = \mathbf{T} \cdot \mathbf{A} \cdot \mathbf{T}^T$ and $\dot{\mathbf{M}} = \mathbf{T} \cdot \mathbf{M} \cdot \mathbf{T}^T$ are embedded in the solenoidal subspace. In order to find a base $\mathbf{T}^T$ for the solenoidal subspace we use the relationship ker $\mathbf{R}^T$ = ker ($\mathbf{R} \cdot \mathbf{R}^R$). The kernel of the operator $\mathbf{R} \cdot \mathbf{R}^\mathbf{T}$ satisfies the following eigenvalue problem:

$$\mathbf{R} \cdot \mathbf{R}^T \mathbf{t} = \eta\, \mathbf{t} \quad (27)$$

for the pairs $\{\mathbf{t}_1, \eta_1 = 0, \ldots, \mathbf{t}_{\dot{n}}, \eta_{\dot{n}} = 0\}$.

5. Numerical results

In this section the solenoidal subspace that can be generated from the quadratic element and its combinations with the linear elements will be analyzed. These elements are: the linear continuous element (P1), the linear continuous element with central node ($P1^+$), and the linear discontinuous element ($P1^D$).

Divergence operators for each one of the combinations of the quadratic element with the linear elements previously nominated are built. Thus, three solenoidal subspaces are obtained, as shown in Fig. 1: quadratic-linear-continuous (J-p1p2), bubble (J-Bu), and quadratic-linear discontinuous (J-dd). It is natural to wonder about the behavior of a generalized problem of eigenvalues in each one of these subspaces. When projecting the same jacobian and mass matrix in each one of these solenoidal subspaces, three generalized eigenvalues problems of different dimensions show up. Table 1 shows the main discretization parameters for a combination of between quadratic elements and the continuous linear with central node $P1^+$ elements. The

Table 1
Parameters of discretizations

Class	Mesh	Elem	dim $(\mathbf{V}_0^h)$	dim$(\mathbf{S}_0^h)$	dim(ker$(\mathbf{R}_h^T)$)
P2P1	1	196	870	120	626
	2	784	3306	435	2635
	3	3136	12882	1653	10769
$P1^+$	1	196	870	316	431
	2	784	3306	1219	1852
	3	3136	12882	4789	7634
$P1^D$	1	196	870	588	160
	2	784	3306	2352	720
	3	3136	12882	9408	3016

first column shows the triangular meshes used in the construction of the divergence operator. In the second column are the number of triangular elements for each mesh, in the third column are the dimensions of the velocity spaces, in the fourth column are the dimensions of the pressure spaces, and in the fifth column are the dimensions of the solenoidal subspaces.

Figure 1 shows in a logarithmic scale the dimension of the solenoidal subspace described in Table 1. It can be observed that the three solenoidal subspaces (discontinuous solenoidal subspace ${R^T}_{P1D}$, bubble solenoidal subspace R^T_{P1+}, and linear-continuous solenoidal subspace R^T_{P2}), grow proportionally to its dimensions and satisfy the relation

$$dim(R^T_{P1^D}) \quad < \quad dim(R^T_{P1^+}) \quad < \quad dim(R^T_{P2}) \qquad (28)$$

Figure 2 shows the stability spectra for the same problem projected in the previously studied subspaces. When observing the spectra of Fig. 2 we see that the topological structure of the dangerous eigenvalue remains invariable. This shows that the solenoidal subspace built from the divergent discontinuous operator catch the dynamic behavior as well as solenoidal subspaces with larger dimensions. For quadratic elements in velocity and linear elements in pressure the calculation can be reduced to 83 percent in degrees of freedom. If we put discontinuous elements in place of the continuous linear ones the calculation can be reduced to 23 percent in degrees of freedom.

6. Conclusion

It is very promising that in a very small subspace it is possible to calculate the stability problem without big modifications in the topology of eigenvalues. This property, which could be called the dynamic subsistence of the discontinuous solenoidal subspace, may have an extraordinary relevance for calculations with a high number of degrees of freedom.

References

[1] Ladyzhenskaia OA. The Mathematical Theory of Viscous Incompressible Flows. New York: Gordon & Breach, 1963.

[2] Teman R. Navier–Stokes equations: Theory and Numerical Analysis. Amsterdam: North Holland, 1977.

[3] Hall CA, Xiu Ye. Construction of null bases for the divergence operator associated with incompressible

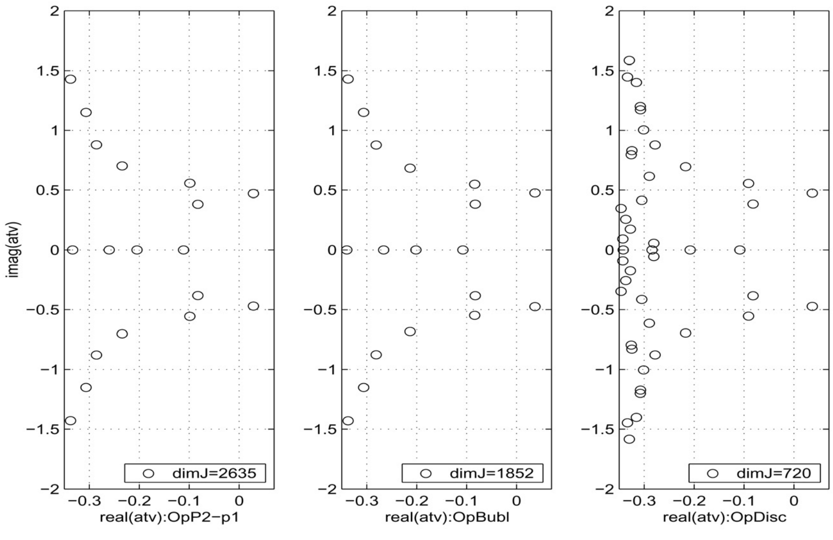

Fig. 2. Stability spectra.

Navier–Stokes equations. Linear Algebra and its Applications 1992;171:9–52.
[4] Bathe KJ. Finite Element Procedures. New Jersey: Prentice Hall, 1996.
[5] Gervais JJ, Lemelin D, Pierre R. Some experiments with stability analysis of discrete incompressible flows in the lid-driven cavity. Int J Numer Methods Fluids 1997;124:477–492.
[6] Brezzi F, Fortin M. Mixed and hybrid finite element methods. Springer Series in Computational Mathematics 1991;15:73–78.
[7] Joseph DD. Hydrodynamic Stability and Bifurcation. New York: Springer-Verlag, 1985.
[8] Fornberg B. Steady viscous flow past a circular cylinder up to Reynolds number 600. J Comput Phys 1985;61:297–320.
[9] Jackson CP. A finite-element study of onset of vortex shedding in flow past variosly shaped bodies. J Fluid Mech 1987;182:23–45.
[10] Noack BR, Eckelmann H. A global stability analisis of the steady and periodic cylinder wake. J Fluid Mech 1994;270:297–330.
[11] Morzynski M, Thiele F. Numerical stability analysis of a flow about a cylinder. Z Angew Math Mech 1991;71:T424–T428.
[12] Ding Y, Kawahara M. Three-dimensional linear stability analysis of incompressible viscous flows using the finite element method. Int J Numer Methods Fluids 1999;31:451–479.
[13] Georgescu A. Hydrodynamic Stability Theory. Boston: Martinus Nijhoff, 1985.
[14] Yudovich VI. On the stability of stationary flow incompressible fluids. Dokl Akad Nauk 1965;161(5):1037–1040.
[15] Prodi G. Teoremi di tipo locale per il sistema di Navier–Stokes e stabilit delle soluzioni stazionarie. Rend Seminario Mat Padova 1962;32:374–397.
[16] Chen JH, Pritchard WG, Tavener SJ. Bifurcation for flow past a cylinder between parallel planes. J Fluid Mech. 1995;284:23–41.
[17] Fortin A, Jardak M, Gerdais JJ, Pierre R. Localization of hopf bifurcations in fluid flow problems. Int J Numer Methods Fluids 1997;24:1185–1210.
[18] Fortin A, Jardak M, Gerdais JJ, Pierre R. Old and new results on the two-dimensional poiseuille flow. J Comput Phys 1994;115:455–469.
[19] Cliffe KA, Graham IG, Scheichl R Stals L. Parallel computation of flow in heterogeneous media modelled by mixed finite elements. J Comput Phys 2000;164:258–282.