Numerical reconstruction of the initial temperature of diapiric structures in the Earth: effect of the heat diffusion

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Abstract

We present a numerical approach to three-dimensional reconstruction of the evolution of thermal diapiric structures (e.g. hot plumes in the Earth's mantle) backwards in time. This approach is based on a search for the temperature and flow in the geological past by minimizing differences between the present-day temperature and that predicted by forward models of the evolution of diapiric structures for an initial temperature guess. We use the Eulerian spline FEM, FDM, and variational method to solve the coupled heat, momentum, and continuity equations with the appropriate boundary and initial conditions. The relevant numerical algorithm is tested with respect to the Rayleigh number and viscosity.

Keywords: 3-D thermal convection; Data assimilation; Variational method; Eulerian spline FEM; FDM

1. Introduction

The idea of combining current and past mantle temperatures and flow data in an explicit dynamical model is based on the variational method and on solving of the coupled heat, momentum, and continuity equations in order to find the model representation that is most consistent with the observations [1,2,3,4]. That best estimate can then be used to analyze geodynamic processes or initialize a convective model setup more accurately.

Here we present the numerical approach to a reconstruction of thermal structures of the Earth, present a model example, and discuss the effects of the Rayleigh number (diffusion) on the efficiency of the reconstruction algorithm.

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2. Mathematical statement of the problems

In 3D model domain, where $\Omega = [0, x_1 = 3h] \times [0, x_2 = 3h] \times [0, x_3 = h]$, where $\mathbf{x} = (x_1, x_2, x_3)$ are the Cartesian coordinates, we consider a viscous convective flow of incompressible fluid (heated from below) at the infinite Prandtl number with a temperature-dependent viscosity. The flow is described by heat, momentum, and continuity equations. In the Boussinesq approximation these dimensionless equations take the form:

$$\partial T/\partial t + \mathbf{u}\cdot\nabla T - \nabla^2 T = 0, \quad t \in (0,\vartheta], \quad \mathbf{x} \in \Omega \tag{1}$$

$$\nabla P = \mathrm{div}(\eta \mathbf{E}) + RaT\mathbf{e}_3, \quad \mathbf{E} = \{\partial u_i/\partial x_j + \partial u_j/\partial x_i\}, \quad \mathbf{e}_3 = (0,0,1) \tag{2}$$

$$\mathrm{div}\,\mathbf{u} = 0, \quad t \in [0,\vartheta], \quad \mathbf{x} \in \Omega \tag{3}$$

where T, t, $\mathbf{u} = (u_1, u_2, u_3)$, P, and η are temperature, time, velocity, pressure, and viscosity, respectively, and Ra is the Rayleigh number.

At the boundary of the model domain we set the impenetrability condition with perfect slip conditions. We assume the heat flux through the vertical boundaries of the box to be zero. The upper and lower boundaries are isothermal surfaces. To solve the direct (forward in time) and inverse (backward in time) problems of thermal convection, we assume the temperature to be known at the initial time $t = 0$ and at the final time $t = \vartheta$, respectively.

3. Variational method and numerical approach

We consider the following objective functional $J(\varphi) = \|T(\vartheta, \cdot;\varphi) - \chi(\cdot)\|^2$, where $T(\vartheta,\cdot;\varphi)$ is the solution of the thermal boundary problem (1) at the final time ϑ, which corresponds to some (unknown as yet) initial temperature distribution $\varphi(\mathbf{x})$; $\chi(\mathbf{x}) = T(\vartheta, \mathbf{x}; T_0)$ is the known temperature distribution at the final time, which corresponds to the initial temperature $T_0 = T_0(\cdot)$; and $\|\cdot\|$ is the norm in space $L^2(\Omega)$. The functional has its unique global minimum at value $\varphi \equiv T_0$ and $J(T_0) \equiv 0$, $\nabla J(T_0) \equiv 0$. Therefore, we seek the global minimum of the functional with respect to φ. To find a minimum of the functional we employ the following gradient method ($k = 0,\ldots, n,\ldots$):

$$\varphi_{k+1} = \varphi_k - \beta_k \nabla J(\varphi_k),$$
$$\beta_k = \min\{1/(k+1), J(\varphi_k)/\|\nabla J(\varphi_k)\|\}, \quad \varphi_0 = T_* \quad (4)$$

where T_* is an initial temperature guess. We found ∇J as a solution to the boundary value problem adjoint to Eq. (1):

$$\partial Z/\partial t + 0.5(\mathbf{u}\cdot\nabla Z + \nabla\cdot(Z\mathbf{u})) + \nabla^2 Z = 0,$$
$$t \in [0,\vartheta), \quad \mathbf{x} \in \Omega, \quad (5)$$
$$Z(\vartheta,\mathbf{x}) = 2(T(\vartheta,\mathbf{x};\varphi) - \chi(\mathbf{x})), \quad \mathbf{x} \in \Omega,$$

supplied with uniform boundary conditions. The iterative solution algorithm for the backward heat equation is based on the following three steps: (i) to solve the heat equation (1) with appropriate boundary conditions and initial condition $T(0, \mathbf{x}) = \varphi_k(\mathbf{x})$ at time interval $[0, \vartheta]$ in order to find $T(\vartheta, \mathbf{x}; \varphi_k)$; (ii) to solve problem (5) backwards in time and to determine $\nabla J(\varphi_k)$; and (iii) to determine β_k and then to update the previous temperature, i.e. to find φ_{k+1} from Eq. (4). Computations are terminated, when $\delta\varphi_n = J(\varphi_n) + \|\nabla J(\varphi_n)\|^2 < \varepsilon$, and φ_n is then considered to be an approximate solution to the backward heat equation.

Temperature entering in the heat equation (1) is approximated by finite differences and found by the alternating direction method. A numerical solution to the momentum equation (2) is based on an introduction of a two-component vector velocity potential and on the application of the Eulerian FEM with a tricubic-spline basis for computing the potential. For more detail of the variational method and numerical approach used, see [2,3,4,5].

At each subinterval of time $[t_{n+1}, t_n]$ ($0 < \ldots < t_{n+1} < t_n < \ldots < t_0 = \vartheta$), the solution of the problem (1)–(3) with the appropriate boundary and initial conditions backwards in time consists of the following steps.

Step 1. Given the temperature $T = T(t_n,\cdot)$ at $t = t_n$ Eqs. (2) and (3) are solved to determine the velocity $\mathbf{u}$.

Step 2. The 'advective' temperature $T_{adv} = T_{adv}(t_{n+1},\cdot)$ is determined by solving the advection heat equation (neglecting the diffusion term) backwards in time, and Steps 1 and 2 are then repeated to find the velocity, $\mathbf{u}_{adv} = \mathbf{u}(t_{n+1}, \cdot; T_{adv})$, corresponding to the 'advective' temperature.

Step 3. The velocities $\mathbf{u}_{adv}$ and $\mathbf{u}$ are used in Eqs. (1) and (5), respectively, to find temperature $T = T(t_{n+1},\cdot)$ at $t = t_{m+1}$.

Step 2 is used to accelerate the convergence of $\delta\varphi_n$ to a prescribed value of ε at a strong flow (high Rayleigh numbers). When the flow is weak, Step 2 is omitted and $\mathbf{u}_{adv}$ is replaced by $\mathbf{u}$. After these algorithmic steps, we obtain temperature $T = T(t_m,\cdot)$ and velocity $\mathbf{u} = \mathbf{u}(t_m,\cdot)$ corresponding to $t = t_n$.

4. Model example and performance of the algorithm

We model the evolution of mantle plumes deprived of source material through numerical experiments of 3D thermal convection in a bottom heated box. Plume deprivation is simulated by imposing sudden reductions in *Ra*. At the initial time we assume a linear temperature stratification in the model. The mantle behaves as a Newtonian fluid at geological time scales, and a temperature-dependent viscosity law given by $\eta = \exp(Q/(T + G) - Q/(0.5 + G))$ is used in the modeling, where $Q = [225/\ln(r)] - 0.25\ln(r)$, $G = 15/\ln(r) - 0.5$, and r is the viscosity ratio between the upper and lower boundaries of the model domain. We run the model for two values of $r = 20$ and $r = 200$. This domain is divided into $37 \times 37 \times 29$ rectangular finite elements to approximate the vector velocity potential by tricubic splines, and a uniform grid $112 \times 112 \times 88$ is employed for approximation of temperature, velocity, and viscosity.

We assume that mantle plumes are generated at the base of the upper mantle boundary by a random temperature perturbation on this boundary. The plumes move upward through the model domain, gradually forming structures with well-developed heads and tails. We interrupt the numerical experiment at a certain time when the hot material in the source layer is nearly depleted and the plume heads are mushroom shaped, and decrease the Rayleigh number. We then continue three independent experiments assigning various *Ra* less than the initial *Ra* by one to three orders of magnitude.

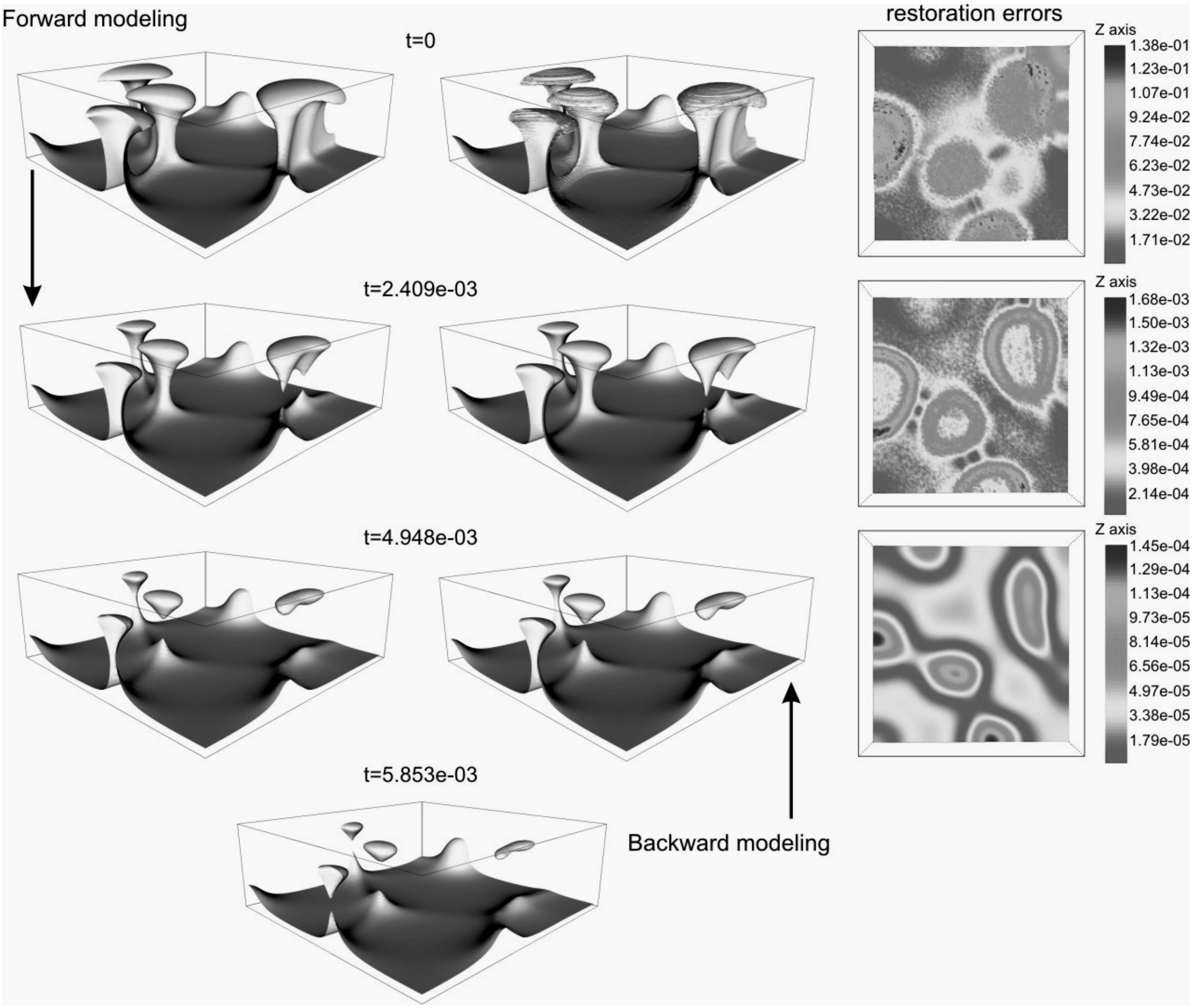

Fig. 1. Reconstruction of thermal plumes in the Earth's upper mantle. The plumes are represented by the isothermal surface at $T = 0.92$.

Here we present the case of $Ra = 9.5 \times 10^3$. Figure 1 (left panel) shows several stages in the diffusive decay of the mantle plumes. The thermal plumes diminish in size with time and the plume tails disappear before the plume heads.

We apply the numerical approach to reconstruct the plumes from their 'present' weak state to the prominent state they were in the past. Figure 1 illustrates the reconstructed states of the plumes (middle panel) and the temperature residuals δT (right panel) between the temperature predicted by the forward model and the temperature reconstructed to the same age.

The performance of the algorithm is evaluated in terms of the number of iterations n required to achieve ϵ, the prescribed relative reduction of $\delta\varphi_n$. Figure 2 shows that the smaller is the Rayleigh number (the higher diffusion), the larger number of iterations is required to achieve ϵ. This figure illustrates that the first 4 to 6 iterations contribute mainly to the reduction of $\delta\varphi_n$.

5. Conclusion

A mathematical model of the thermal plume evolution in the Earth's mantle is described by a set of equations, and we have demonstrated that the set of equations can be solved numerically backwards in time. Our restoration methodology works well for the mathematical model, and we have shown its efficiency in the framework of this model. We have illustrated that a weak (diffused) present-day thermal feature in the mantle can

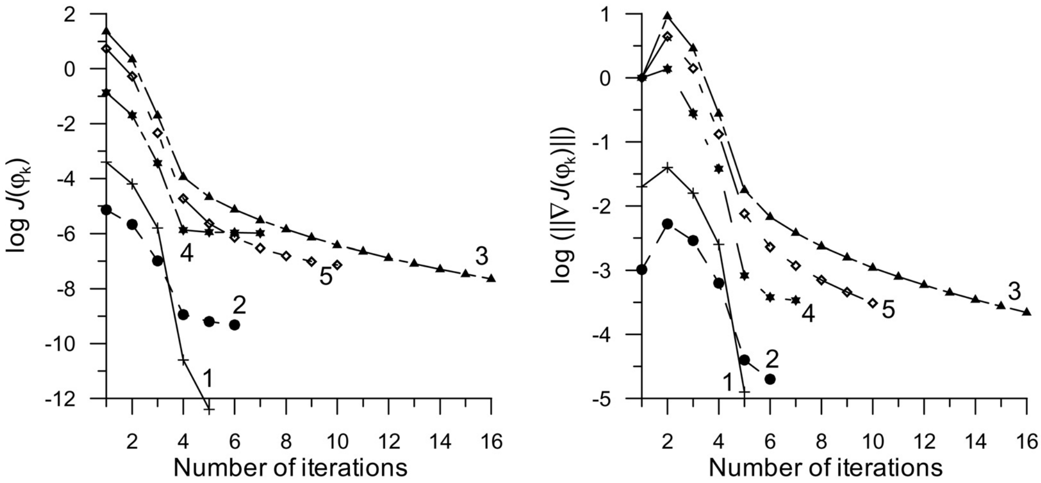

Fig. 2. Relative reductions of the objective functional J (left panel) and the norm of the gradient of J (right panel) as functions of the number of iterations. Curves: $r = 20$ and $Ra = 9.5 \times 10^5$ (*1*), $Ra = 9.5 \times 10^3$ (*2*), $Ra = 9.5 \times 10^2$ (*3*); $r = 200$ and $Ra = 9.5 \times 10^3$ (*4*); $Ra = 9.5 \times 10^2$ (*5*).

be traced back into the geological past and the structure prominent in this past can be restored.

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