Toward a definition of LES

J.-L. Guermond[a,*], S. Prudhomme[b]

[a]Texas A&M University, Mathematical Department, College Station, TX 77843, USA
[b]ICES, University of Texas at Austin, Austin, TX 78712, USA

Abstract

A constructive definition of LES is proposed and illustrated with examples.

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1. Introduction

During the past 40 years a new class of turbulence models, collectively known as Large Eddy Simulation (LES) models [1], has emerged in the literature. An extended variety of LES models is now available but no satisfactory mathematical theory for LES has yet been proposed. More surprisingly, no mathematical definition of LES has been stated either, although some qualitative attempts have been made in this direction. The objective of the present work is to go beyond qualitative statements by proposing a coherent mathematical definition of LES approximations of the Navier–Stokes equations.

2. The definition

Let Ω be the bounded fluid domain in $\mathbb{R}^3$. Let $\mathbf{X}$ be a closed subspace of $\mathbf{H}^1(\Omega)$ and M be a closed subspace of $L^2(\Omega)$ such that the Navier–Stokes equations

$$\partial_t\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} + \nabla p - \nu\nabla^2\mathbf{u} = \mathbf{f}, \quad \nabla\cdot\mathbf{u} = 0, \quad \mathbf{u}_{|t=0} = \mathbf{u}_0, \quad +\text{B.C.} \tag{1}$$

have a weak solution $(\mathbf{u}, p)$ in the Leray class, i.e. $\mathbf{u} \in L^\infty(0, T; \mathbf{L}^2(\Omega)) \cap L^2(0, T; \mathbf{X})$ and $p \in \mathcal{D}'(]0, T[, L^2(\Omega))$. Since we shall use the concept of suitable weak solution, we recall the following:

Definition 1 *A weak solution to the Navier–Stokes equation* $(\mathbf{u}, p)$ *is suitable if* $\mathrm{u} \in L^2(0,T;\mathbf{X}) \cap L^\infty(0,T;\mathbf{L}^2(\Omega)), \{p \in L^{\frac{3}{2}}(]0,T[\times \mathrm{M})$ *and the local energy balance* $\partial_t(\frac{1}{2}\mathbf{u}^2) + \nabla\cdot((\frac{1}{2}\mathbf{u}^2 + \mathrm{p})\mathbf{u}) - \nu\nabla^2(\frac{1}{2}\mathbf{u}^2) + \nu(\nabla\mathbf{u})^2 - \mathbf{f}\cdot\mathbf{u} \le 0$ *is satisfied in the distributional sense.*

Then, we propose the following:

Definition 2 *A sequence* $(\mathbf{u}_\gamma, p_\gamma)_{\gamma>0}$ *with* $\mathbf{u}_\gamma \in L^\infty(0, T; \mathbf{L}^2(\Omega)) \cap L^2(0,T;\mathbf{X})$ *and* $p_\gamma \in \mathcal{D}'(]0, T[; L^2(\Omega))$ *is said to be a LES approximation to the Navier–Stokes equations if*

(i) *There are two finite-dimensional vectors spaces* $\mathbf{X}_\gamma \subset \mathbf{X}$ *and* $M_\gamma \subset L^2(\Omega)$ *such that* $\mathbf{u}_\gamma \in \mathcal{C}^1([0, T]; \mathbf{X}_\gamma)$ *and* $p_\gamma \in \mathcal{C}^0([0, T]; M_\gamma)$ *for all* $T > 0$.
(ii) *The sequence converges (up to subsequences) to a weak solution of the Navier–Stokes equations, say* $\mathbf{u}_\gamma \to \mathbf{u}$ *weakly in* $L^2(0, T; \mathbf{X})$ *and* $p_\gamma \to p$ *in* $\mathcal{D}'(]0, T[; L^2(\Omega))$.
(iii) *The weak solution* $(\mathbf{u}, p)$ *is suitable.*

In practice the construction of a LES model is decomposed into three steps: (1) Construction of a pre-LES model: This step consists of regularizing the NS equations by introducing a parameter ε representing the large eddy scale beyond which the nonlinear effects are dampened. The purpose of the regularization process is to yield a well-posed problem for all times. Moreover, the limit solution of the pre-LES model must be a weak solution to the NS equations as $\varepsilon \to 0$ and should be suitable. (2) Discretization of the pre-LES model: This step introduces the meshsize parameter h and the finite-dimensional spaces $\mathbf{X}_\gamma$, M_γ. (3) Determination of a (possibly maximal) relationship between ϵ and h: The parameters ϵ and h should be selected in such a way that the discrete solution is ensured to converge to a suitable solution of the NS equations when $\varepsilon \to 0$ and $h \to 0$.

We now briefly illustrate the definition by going through two examples.

* Corresponding author. Tel.: +1 (979) 862 4890; Fax: +1 (979) 862 4190; E-mail: Guermond@math.tamu.edu

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3. Hyperviscosity model

Lions [2] proposed the following hyperviscosity perturbation of Eq. (1):

$$\begin{cases} \partial_t \mathbf{u}_\varepsilon + \mathbf{u}_\varepsilon \cdot \nabla \mathbf{u}_\varepsilon + \nabla p_\varepsilon - \nu \nabla^2 \mathbf{u}_\varepsilon + \varepsilon^{2\alpha}(-\nabla^2)^\alpha \mathbf{u}_\varepsilon = \mathbf{f} \\ \quad \text{in } \Omega\times]0,T[, \\ \nabla \cdot \mathbf{u}_\varepsilon = 0 \quad \text{in } \Omega\times]0,T[, \\ \mathbf{u}_\varepsilon|_\Gamma, \partial_n \mathbf{u}_\varepsilon|_\Gamma, \ldots, \partial_n^{\alpha-1}\mathbf{u}_\varepsilon|_\Gamma = 0, \quad \text{or } u_\varepsilon \text{ is periodic}, \\ \quad \mathbf{u}|_{t=0} = \mathbf{u}_0 \end{cases} \tag{2}$$

where $\varepsilon > 0$ and α is an integer. The appealing aspect of this perturbation is that it yields a well-posed problem in the classical sense when $\alpha \geq \frac{5}{4}$ in three space dimensions and the limit solution as $\varepsilon \to 0$ is suitable; hence the above hyperviscosity model is a pre-LES model as defined in the previous section.

We now introduce the corresponding LES model. For the sake of simplicity, we restrict ourselves to periodic boundary conditions and spectral approximation techniques, but we stress that Definition 2 is not restricted to this simplified setting.

Let $\mathbb{P}_N = \{p(\mathrm{x}) = \sum_{|\mathrm{k}|_\infty \leq N} c_\mathrm{k} e^{i\mathrm{k}\cdot\mathrm{x}}, c_\mathrm{k} = \bar{c}_{-\mathrm{k}}\}$ be the set of real-valued trigonometric polynomials of partial degree less than or equal to N on Ω. To approximate the velocity and the pressure fields we introduce the following finite-dimensional vector spaces:

$$\mathbf{X}_N = \dot{\mathbb{P}}_N^3, \quad \text{and} \quad M_N = \dot{\mathbb{P}}_N \tag{3}$$

We start by introducing the hyperviscosity parameter $\frac{5}{4}$. We then introduce θ, with $0 < \theta < 1$, from which we define the large scale parameter $N_i = N^\theta$ and the corresponding large eddy cutoff scale $\varepsilon_N = {}^1/_{N_i}$. We also introduce a hyperviscosity kernel $\mathrm{Q}(\mathbf{x}) = (2\pi)^{-3} \sum_{N_i \leq |\mathbf{k}|\infty \leq N} |\mathbf{k}|^{2\alpha_e i\mathrm{k}\cdot\mathrm{x}}$. When α is an integer, $Q*(\cdot)$ is the α-th power of the restriction of the Laplace operator on the space spanned by the Fourier modes comprised between N_i and N. Note that when θ increases, the vanishing viscosity amplitude decreases and the range of wavenumbers on which the kernel $Q(\mathbf{x})$ is active shrinks.

The LES approximation consists of the following:

$$\begin{cases} \text{Seek } \mathbf{u}_N \in \mathcal{C}^1([0,T];\ \mathbf{X}_N) \text{ and } p_N \in \mathcal{C}^0([0,T];\ M_N) \\ \quad \text{such that} \\ (\partial_t \mathbf{u}_N, \mathbf{v}) + (\mathbf{u}_N \cdot \nabla \mathbf{u}_N, \mathbf{v}) - (p_N, \nabla \cdot \mathbf{v}) + \nu(\nabla \mathbf{u}_N, \nabla \mathbf{v}) \\ \qquad + \varepsilon_N^{2\alpha}(Q * \mathbf{u}_N, \mathbf{v}) = (\mathbf{f}, \mathbf{v}), \quad \forall \mathbf{v} \in \mathbf{X}_N, \forall t \in (0,T], \\ (\nabla \cdot \mathbf{u}_N, q) = 0, \quad \forall q \in M_N, \forall t \in (0,T], \\ \quad (\mathbf{u}_N, \mathbf{v})|_{t=0} = (\mathbf{u}_0, \mathbf{v}), \quad \forall \mathbf{v} \in \mathbf{X}_N \end{cases} \tag{4}$$

The main result of this section is the following:

Theorem 1 *Let* $\mathbf{f} \in L^2(0, T; \mathbf{L}^2(\Omega))$ *and* $\mathbf{u}_0 \in \mathbf{H}^\alpha(\Omega) \cap \mathbf{V}$. *Assume that*

$$0 < \theta < \frac{4\alpha - 5}{4\alpha} \quad \textit{if } \alpha \leq \frac{3}{2}, \quad \textit{or} \quad 0 < \theta < \frac{2(\alpha-1)}{2\alpha+3} \quad \textit{otherwise} \tag{5}$$

then, up to subsequences, u_N *converges weakly in* $L^2(0, T; \mathbf{X})$ *and strongly in any* $L^r(0, T; \mathbf{L}^s(\Omega))$, *with* $2 \leq \mathrm{r} < \frac{4\mathrm{s}}{3(\mathrm{s}-2)} < +\infty$ *to a suitable solution of Eq. (1) as N goes to infinity.*

With the polynomial degree N we can associate the mesh size $h_N = N^{-1}$. Then we observe that Eq. (5) means that $N^{-1} = h_N \ll \varepsilon_N = N^{-\theta}$. In other words, the large eddy scales must be significantly larger than the grid size for the limit solution to be suitable.

4. The discrete Leray-α model

Still assuming periodic boundary conditions, the second pre-LES model we consider is the so-called Leray-α model [3]:

$$\begin{cases} \partial_t \mathbf{u}_\varepsilon + \overline{\mathbf{u}_\varepsilon} \cdot \nabla \mathbf{u}_\varepsilon - \nu \nabla^2 \mathbf{u}_\varepsilon + \nabla p_\varepsilon = \mathbf{f}, \\ \overline{\mathbf{u}_\varepsilon} - \varepsilon^2 \nabla^2 \overline{\mathbf{u}_\varepsilon} = \mathbf{u}_\varepsilon, \qquad \nabla \cdot \mathbf{u}_\varepsilon = 0, \\ \mathbf{u}_\varepsilon, \text{ and } \overline{\mathbf{u}_\varepsilon} \text{ are periodic}, \qquad \mathbf{u}_\varepsilon|_{t=0} = \mathbf{u}_0. \end{cases} \tag{6}$$

The couple $(\mathbf{u}_\epsilon, p_\epsilon)$ can be shown to converge to a suitable solution of Eq. (1) when $\varepsilon \to 0$.

Still keeping the Fourier framework, we now consider the corresponding LES model:

$$\begin{cases} \text{Seek } \mathbf{u}_N \in \mathcal{C}^1([0,T]; \mathbf{X}_N) \text{ and } p_N \in \mathcal{C}^0([0,T]; M_N) \\ \quad \text{such that} \\ \text{for all } t \in (0,T], \text{ for all } \mathbf{v} \in \mathbf{X}_N, \text{ and for all } q \in M_N, \\ (\partial_t \mathbf{u}_N, \mathbf{v}) + (\bar{\mathbf{u}}_N \cdot \nabla \mathbf{u}_N, \mathbf{v}) - (p_N, \nabla \cdot \mathbf{v}) + \nu(\nabla \mathbf{u}_N, \nabla \mathbf{v}) = \\ \quad (\mathbf{f}, \mathbf{v}), \\ (\bar{\mathbf{u}}_N, \mathbf{v}) + \varepsilon_N^2(\nabla \bar{\mathbf{u}}_N, \nabla \mathbf{v}) = (\mathbf{u}_N, \mathbf{v}), \qquad (\nabla \cdot \mathbf{u}_N, q) = 0, \\ \quad (\mathbf{u}_N, \mathbf{v})|_{t=0} = (\mathbf{u}_0, \mathbf{v}) \end{cases} \tag{7}$$

where, upon chosing $0 < \theta < 1$, $\varepsilon_N = N_i^{-1} = N^{-\theta}$ is the scale of the smallest eddies that we authorize to be nonlinearly active, i.e. ε_N is the large eddy scale.

Theorem 2 *Let* $\mathbf{f} \in L^2(0, T; \mathbf{L}^2(\Omega))$ *and* $\mathbf{u}_0 \in \mathbf{H}$. *Assume that* $\theta < \frac{2}{3}$, *then* $\mathbf{u}_N$ *converges weakly, up to subsequences, in* $L^2(0, T; \mathbf{X})$ *and strongly in any* $L^r(0, T; \mathbf{L}^s(\Omega))$, *with* $2 \leq r < \frac{4s}{3(s-2)} < +\infty$, *to a suitable solution of (1) as N goes to infinity.*

Denoting by $h_N = N^{-1}$ the discretization scale, Theorem 2 shows that if $\varepsilon_N^{\frac{3}{2}} \gg h_N$, then the pair $(\mathbf{u}_N, p_N)$ is a LES approximation in the sense of Definition 2.

5. Conclusions

A definition for LES approximations of the Navier–Stokes equations has been proposed. This definition introduces two parameters, i.e. a discretization scale h

and a large eddy scale ε. But first and foremost, the definition gives a rule for choosing the ratio h/ε, i.e. the limit solution as $h \to 0$ and $\varepsilon \to 0$ must be suitable. We have applied this rule on two examples and we have shown in both cases that h should be chosen much smaller than ε. This result clearly challenges what is often suggested in the literature and commonly done in practice, namely to take ε of the same order as h.

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