On the application of symmetry methods for turbulence modelling

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Abstract

Since the symmetries of fluid motion (scaling of time, scaling of space, translation in time, translation in space, Galilean invariance etc.) are admitted by all statistical quantities of turbulent flows as can be taken from the multi-point equations [1], we can derive conditions for turbulence models so that they capture the proper flow physics. Concerning these constraints we will investigate two-equation models as well as Reynolds stress transport models for their capability to reproduce the new exponential velocity law first derived in Oberlack [1].

Keywords: Turbulent scaling laws; Lie symmetries; Exponential velocity law; Statistical turbulence models; Model calibration

1. Introduction

In RANS modeling it is common practice to use classical canonical flow cases such as the isotropic decay, the logarithmic law of the wall or homogeneous shear flows for calibrating the model constants. With the help of Lie group analysis a broad variety of invariant solutions (scaling laws) can be derived comprising the classical solutions of the latter (the isotropic decay, logarithmic law of the wall, homogeneous shear flow) as well as a broad variety of new solutions that have so far not been used for model calibration or development. The symmetry methods provide therefore a very useful tool for the improvement of existing turbulence models or may be a guideline for the development of new models.

2. Required symmetry conditions for turbulence models

The most important necessary conditions for Reynolds-averaged turbulence models may be summarized as follows:

(i) All symmetries of the two- and multi-point correlation equations have to be admitted by the model equations (necessary but not sufficient condition!).
(ii) There should also be no additional unphysical symmetries in the model equations for reduced cases such as those admitting rotational symmetry.
(iii) The symmetry conditions (i) and (ii) have to be admitted from each single model equation and independent of the momentum and continuity equations.
(iv) All invariant solutions implied by the two- and multi-point correlation equations also have to be admitted by the model equations.

Condition (ii) emerged from symmetry analyses of the κ–ϵ model in plane and axisymmetric parallel shear flows with rotation. From these test cases we found that the κ–ϵ model has too many symmetries that are not contained in the two- and multi-point equations, leading to non-physical behavior under certain flow conditions, such as rotation or stream line curvature. This is due to the fact that the κ–ϵ model does not contain Coriolis terms for any type of flow so that no symmetry breaking of scaling of time is allowed. A complete group analysis of the κ–ϵ equations in cylinder coordinates discloses an additional symmetry of the form:

$$r^* = r, \quad \bar{u}_z^* = \bar{u}_z, \quad \bar{u}_\phi^* = \bar{u}_\phi + br, \quad \epsilon^* = \epsilon \tag{1}$$

where b represents the group parameter. This additional symmetry allows the addition of solid body rotation to the azimuthal velocity without any change to the remaining flow quantities. Obviously this is unphysical since turbulence is highly sensitive to rotation.

Khor'kova et al. [2] found from a symmetry analysis of the κ–ϵ model that condition (i) is usually fulfilled by

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the model equations. Though the κ–ϵ model apparently admits all necessary symmetries, we find that it is still not capable of reproducing all invariant solutions that are derived from the symmetries (condition (iv)). A first hint towards this problem is given in [3] and [4] investigating shear-free turbulent diffusion. This clear contradiction may be illuminated by the example of the exponential velocity law for the zero-pressure gradient (ZPG) turbulent boundary layer flow (see Section 3).

3. Consistency of RANS models concerning the modelling of the exponential velocity defect region

3.1. Symmetry analysis

Analyzing the multi-point-correlation equations for parallel turbulent shear flows, and ZPG turbulent boundary layer flows, a new exponential law has been found, which was identified as an explicit analytic form of the velocity defect law. Scaling the wall-normal coordinate in the outer region with the Clauser-Rotta length scale ($\Delta = \frac{\delta_* \bar{u}_\infty}{u_\tau}$, where δ_* is the displacement thickness) and applying the free-stream boundary condition, the exponential law can be written in general form as

$$\frac{\bar{u}_\infty - \bar{u}_1}{u_r} = F(\eta) = \alpha \exp(-\beta\eta) \tag{2}$$

where α and β are universal constants and $\eta = y/\Delta$. In Oberlack [1] the exponential law (2) was solely derived from the Lie symmetries of the Navier-Stokes equation and in turn from the multi-point correlation equations, introducing the assumption of symmetry breaking of the scaling of space.

3.2. Results from experiments and numerical simulations

Recently the theory has been carefully tested against very high quality experimental data from the KTH database [5] and the Illinois windtunnel [6]. It can be shown that the exponential law fits the experimental data very well in the range of about $0.025 \leq y/\Delta \leq 0.11$, as can be seen from Fig. 1. The constants are determined as $\alpha = 10.5$ and $\beta = 9.5$.

Figure 2 shows the results of a direct numerical simulation (DNS) of a ZPG boundary layer performed for $R_\theta = 2240$ [7]. The DNS results show an exponential law in the region $0.025 \leq y/\Delta \leq 0.15$.

3.3. Model implications

Since a very good agreement between theory, experiments and numerical simulations is observed it should also be demanded from the RANS models to be in accordance with the theory. Thus a further symmetry analysis of the κ–ϵ model for ZPG boundary layer flows has been performed. Imposing here as well the symmetry breaking of the scaling of space we obtain besides Eq. (2) the following set of invariant solutions:

$$k = C \exp(-2\beta\eta), \ \epsilon = D \ \exp(-3\beta\eta) \tag{3}$$

The model equations thus formally admit all symmetries of the correlation equations (condition (i)). Therefore, it remains to check if they also admit the invariant solutions (condition (iv)). The models that have been tested concerning this requirement are the one-equation model from Spalart et al. [8], the classical κ–ϵ model from Hanjalić et al. [9], the κ–ω model from Wilcox [10], the

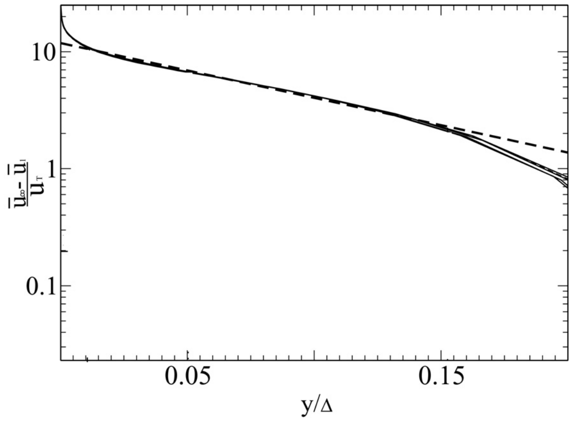

Fig. 1. Mean velocity profiles from experiments at different Reynolds numbers: $Re_\theta = 22579, 23309, 23870, 25767, 26612, 27320$, performed at KTH (Stockholm) [7]; - - - exponential law.

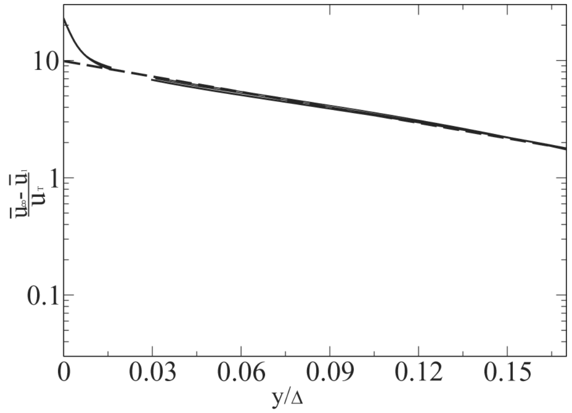

Fig. 2. Mean velocity profiles from DNS with $Re_\theta = 2240$, [5]; - - - exponential law.

v^2f model from Durbin [11], the *SST* model from Menter [12], the $\sqrt{\mathrm{k}}\mathrm{L}$ model from Menter et al. [13], the κ–κL model from Rotta [14], and, as an example for a Reynold stress model, the *LRR* model [15]. Thereby it was found that all these models are in contradiction to the theory. In the following examples we will point out the problems appearing in the κ–ϵ, *Spalart-Almaras* and *LRR* models. For our investigations we assumed that all statistical quantities depend only on the wallnormal coordinate. In a first analysis the diffusion term has been neglected, supposing a local equilibrium between production and dissipation (see Fig. 3). Introducing the invariant solutions (2) and (3) into these simplified model equations, we receive either conditions for the model constants such as $C_{\epsilon 1} = C_{\epsilon 2}$ that are not fulfilled or unphysical coefficients for the exponential law. Replacing the dependent variable $\tilde{\nu}$ in the *Spalart-Almaras* model with the invariant solution $\tilde{\nu} = E \exp(-\beta y)$, we receive the coefficient $E = -C_{\mathrm{b}1}\sigma\alpha \mathrm{u}_\tau \Delta/(2\mathrm{C}_{\mathrm{b}2}\beta)$, whereby C_{b1}, σ and C_{b2} are model constants. We can thus derive the condition $C_{b1}\sigma/C_{b2} < 0$, under which a proper modeling of the exponential region is assured. Since C_{b1}, σ and C_{b2} are positive, changing the algebraic sign of one of the coefficients would probably lead to a deficient modeling if other flow cases are considered. For the *LRR* model, such a discrepancy appears in the model constants that all coefficients of the scaling laws become zero if the invariant solutions are introduced into the model equations. Performing a second analysis with the diffusion term included into the model equations did not lead to any improvements concerning these shortcomings.

The reason for these mismatches seems to be due to the fact that the given models are all calibrated employing the classical flow cases. A calibration of the models using symmetry methods would probably improve the described shortcomings.

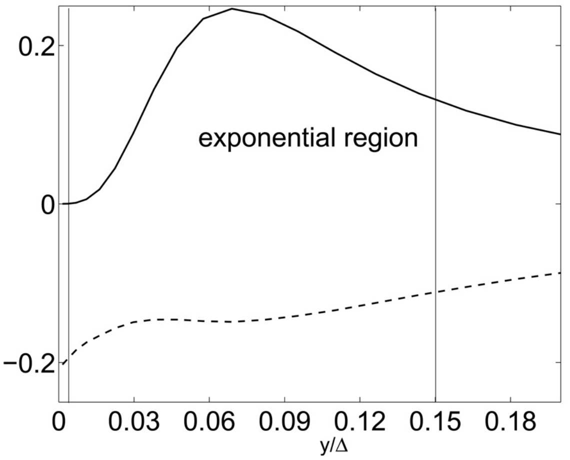

Fig. 3. The turbulent-kinetic-energy budget in a turbulent boundary layer at $Re_\theta = 1410$ [16]; —— production, - - - dissipation.

4. Conclusions and outlook

Using the new exponential velocity law it could be shown that for a proper modeling it is not sufficient that the model equations admit all symmetries of the two- and multi-point correlation equations. We derived therefore as further condition for RANS models that the scaling laws derived from the symmetries also have to be reproduced by the models. From the example of the exponential velocity law we can further propose conditions for the model constants that have to be fulfilled so that the model equations admit the invariant solutions.

It could thus be shown that a considerable improvement in the field of turbulence modeling can be received if symmetry methods are used for the calibration or development of turbulence models.

In the final paper, further investigations concerning condition (ii) will be included as well. We will thus give a full analysis of the κ–ϵ model in axisymmetric parallel shear flows with rotation, as well as proposals for an improvement of classical two-equation models concerning the modeling of this flow case.

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