A two-grid finite volume method for variational multiscale large eddy simulation of turbulent flows

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Abstract

A general class of scale-separating operators based on combined multigrid operators in a two-grid procedure and suited for variational multiscale large eddy simulation is proposed in this work. By applying these scale-separating operators, the complete range of resolved scales is separated into large and small resolved scales. Dynamic as well as constant-coefficient-based subgrid-scale modeling may be performed within this multiscale environment to account for still unresolved scales.

Keywords: Turbulent flow; Large eddy simulation; Finite volume method; Variational multiscale method; Scale separation

1. Introduction

Large eddy simulation (LES) is widely considered to be a promising approach for the numerical simulation of turbulent flows. A basic ingredient of this procedure consists in the separation of resolved and unresolved scales. In classical LES, this separation of scales is mostly achieved by applying a spatial filter. The concept of the variational multiscale method (see, e.g., Hughes et al. [1]) consists in differentiating scale groups through variational projection. In Collis [2] and Gravemeier et al. [3], the picture of the variational multiscale method has been broadened by raising the number of separated scale ranges to three. A three-scale separation may particularly account for large resolved scales, small resolved scales, and unresolved scales. Apart from replacing a spatial filter by variational projection, the (direct) influence of the subgrid-scale model is confined to the small resolved scales in variational multiscale LES. Thus, the large resolved scales are solved without any (direct) influence of the modeling term.

At this stage, it has to be pointed out that the variational multiscale method is, from a practical standpoint, 'merely' a theoretical framework for the separation of scales. Corresponding practical methods in physical space fitting in this framework, on the one hand, and enabling an implementation as a computational algorithm, on the other hand, are still rare. It is important for such practical methods that a clear separation of the different scale ranges is actually achieved. The proposed scale-separating operators of this work have been implemented into the code CDP-α, the flagship LES code of the Center for Turbulence Research. Underlying this code is a finite volume method particularly suited for applications on unstructured grids. With regard to such a computational environment, the procedure for separating the scales is developed. A general class of scale-separating operators based on combined multigrid operators in a two-grid procedure is proposed in order to replace spatial filters or their discrete analogs, respectively, which are widely used in classical LES. One particular representative of this class has the important property of a projector. A projector of this type has also recently been addressed in Koobus et al. [4] as well as Vreman [5]. Further analysis of the scale-separating operators to be presented in this paper comparing them to discrete smooth filters for unstructured grid applications as well as the results for the test case of a turbulent channel flow may be found in Gravemeier [6].

The remainder of this paper is organized as follows. In Section 2, the variational three-scale formulation underlying the variational multiscale LES is described. Section 3 introduces the scale-separating operators. Subgrid-scale modeling within the multiscale environment is

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briefly addressed in Section 4. Finally, some conclusions are drawn in Section 5.

2. Variational three-scale formulation

A variational formulation of the incompressible Navier-Stokes equations reads

$$B_{NS}(\mathbf{v}, q;\ \mathbf{u}, p) = (\mathbf{v}, \mathbf{f})_\Omega \qquad \forall(\mathbf{v}, q) \in \mathcal{V}_{\mathbf{u}p} \tag{1}$$

where $\mathcal{V}_{\mathbf{u}p}$ denotes the combined form of the weighting function spaces for velocity $\mathbf{u}$ and pressure p in the sense that $\mathcal{V}_{\mathbf{u}p} := \mathcal{V}_\mathbf{u} \times \mathcal{V}_p$. The scales of the problem are now separated into three scale ranges as proposed in Collis [2] and Gravemeier et al. [3]. More precisely, it will be dealt with large resolved scales, small resolved scales, and unresolved scales in the following. Due to the linearity of the weighting functions, the variational equation (1) may be decomposed into a system of three variational equations reading

$$B_{NS}(\bar{\mathbf{v}}, \bar{q};\ \bar{\mathbf{u}} + \mathbf{u}' + \hat{\mathbf{u}}, \bar{p} + p' + \hat{p}) = (\bar{\mathbf{v}}, \mathbf{f})_\Omega \quad \forall(\bar{\mathbf{v}}, \bar{q}) \in \overline{\mathcal{V}}_{\mathbf{u}p} \tag{2}$$

$$B_{NS}(\mathbf{v}', q';\ \bar{\mathbf{u}} + \mathbf{u}' + \hat{\mathbf{u}}, \bar{p} + p' + \hat{p}) = (\mathbf{v}', \mathbf{f})_\Omega \quad \forall(\mathbf{v}', q') \in \mathcal{V}'_{\mathbf{u}p} \tag{3}$$

$$B_{NS}(\hat{\mathbf{v}}, \hat{q};\ \bar{\mathbf{u}} + \mathbf{u}' + \hat{\mathbf{u}}, \bar{p} + p' + \hat{p}) = (\hat{\mathbf{v}}, \mathbf{f})_\Omega \quad \forall(\hat{\mathbf{v}}, \hat{q}) \in \widehat{\mathcal{V}}_{\mathbf{u}p} \tag{4}$$

Assuming a clear separation of the large-scale space and the space of unresolved scales and modeling rather than solving for the effect of the unresolved scales results in a model term only acting onto the small resolved scales. The focus is on the subgrid viscosity approach here as a usual and well-established way of taking into account the effect of unresolved scales in classical LES. It has to be emphasized that it is merely accounted for the dissipative effect of the unresolved scales onto the resolved scales by using this approach. The small-scale equation (3) then reads

$$B_{NS}(\mathbf{v}', q';\ \bar{\mathbf{u}} + \mathbf{u}', \bar{p} + p') - (\mathbf{v}', \nu^T \Delta \mathbf{u}')_\Omega = (\mathbf{v}', \mathbf{f})_\Omega \quad \forall(\mathbf{v}', q') \in \mathcal{V}'_{\mathbf{u}p} \tag{5}$$

The subgrid visosity term directly acts only on the small resolved scales. The indirect influence on the large resolved scales, however, is ensured due to the coupling of the large- and the small-scale equation. Appropriate modeling approaches for the subgrid viscosity ν^T will be addressed in Section 4.

The scale separation to be presented in Section 3 relies on a level of complete resolution indicated by the characteristic control volume length h. In terms of the velocity, this reads

$$\mathbf{u}^h = (\ \bar{\mathbf{u}} + \mathbf{u}')^h \tag{6}$$

With respect to this complete resolution level, a large-scale resolution level is identified a priori. This level is characterized by the control volume length $\bar{h}$ and, accordingly, yields a large-scale velocity $\bar{\mathbf{u}}^{\bar{h}}$. The small-scale velocity is consistently defined as

$$\mathbf{u}'^h = \mathbf{u}^h - \bar{\mathbf{u}}^h \tag{7}$$

Large- and small-scale weighting functions are introduced accordingly. It is focussed on a finite volume formulation below, i.e. the subgrid viscosity term is integrated by parts and formulated on the boundary. Reunifying the large-scale equation (2) (without the dependence on the unresolved scales) and the modeled small-scale equation (5) yields a final equation, which may be written in compact form with the help of (7) as

$$\begin{aligned} &B_{NS}\left(\mathbf{v}^h, q^h;\ \mathbf{u}^h, p^h\right) - \left(\mathbf{v}'^h, \nu^T \mathbf{n} \cdot \nabla \mathbf{u}'^h\right)_{\Gamma'} \\ &= B_{NS}\left(\mathbf{v}^h, q^h;\ \mathbf{u}^h, p^h\right) - \left(\mathbf{v}^h, \nu^T \mathbf{n} \cdot \nabla\left(\mathbf{u}^h - \bar{\mathbf{u}}^{\bar{h}}\right)\right)_\Gamma + \\ &\left(\bar{\mathbf{v}}^{\bar{h}}, \nu^T \mathbf{n} \cdot \nabla\left(\mathbf{u}^h - \bar{\mathbf{u}}^h\right)\right)_{\bar{\Gamma}} \\ &= \left(\mathbf{v}^h,\ \mathbf{f}\right)_\Omega \qquad \forall\left(\mathbf{v}^h, q^h\right) \in \mathcal{V}^h_{\mathbf{u}p}; \quad \bar{\mathbf{v}}^{\bar{h}} \in \overline{\mathcal{V}}^{\bar{h}}_{\mathbf{u}p} \end{aligned} \tag{8}$$

where the boundary Γ is split up into a large-scale boundary $\bar{\Gamma}$ and, accordingly, a small-scale boundary $\Gamma' = \Gamma - \bar{\Gamma}$. A visual impression of these boundaries will be given at the end of the subsequent section. The inherent scale separation remains obvious in (8) merely due to the subgrid viscosity term.

3. Separation of scales

As the basis for the following developments, two grids are created: a coarser grid, which is called 'parent' grid, and a finer grid, which is named 'child' grid. The child grid is obtained by an isotropic hierarchical subdivision procedure similar to the one described in Mavriplis [7] starting from the parent grid.

The general class of scale-separating operators based on multigrid operators reads

$$\bar{\mathbf{u}}^h = S^m\left[\mathbf{u}^h\right] = P \circ R[\mathbf{u}^h] = P\left[\ \bar{\mathbf{u}}^{\bar{h}}\right] \tag{9}$$

where the scale-separating operator S^m is constituted by the sequential application of a restriction operator R and a prolongation operator P. Applying the restriction operator on $\mathbf{u}^h$ yields a large-scale velocity $\bar{\mathbf{u}}^h$ defined at the degrees of freedom of the parent grid which is then prolongated in order to obtain a large-scale velocity $\bar{\mathbf{u}}^{\bar{h}}$ defined at the degrees of freedom of the child grid eventually. Various restriction as well as prolongation

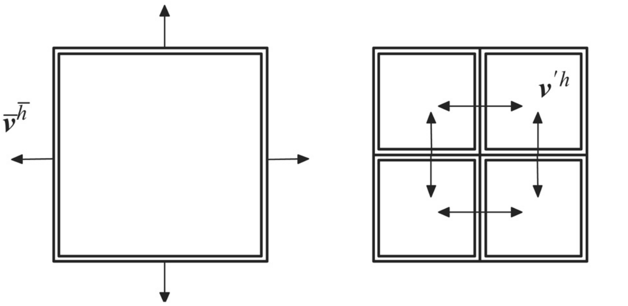

Fig. 1. Geometrical locations of weighting functions in the FVM for a 2-D case: (a) large-scale; (b) small-scale.

operators may be thought of being used in (9). However, the attention is directed to two particular combinations of restriction and prolongation operators. Both of them rely on the same restriction operator, but apply different prolongation operators afterwards. The restriction operator is defined to be a volume-weighted average over all child control volumes Ω_i within one parent control volume subject to

$$\bar{\mathbf{u}}_j^{\bar{h}} = \frac{\sum_{i=1}^{n_{cop}} |\Omega_i| \mathbf{u}_i^h}{\sum_{i=1}^{n_{cop}} |\Omega_i|} \qquad (10)$$

where $\bar{\mathbf{u}}_j^{\bar{h}}$ denotes the large-scale velocity at the center of the parent control volume $\bar{\Omega}_j$ and n_{cop} the number of child control volumes in $\bar{\Omega}_j$. The first prolongation operator P^p yields a constant prolongation, i.e.

$$\bar{\mathbf{u}}_i^h = P^p \left[\bar{\mathbf{u}}_j^{\bar{h}}\right]_i = \bar{\mathbf{u}}_j^{\bar{h}} \qquad \forall \Omega_i \subset \bar{\Omega}_j \qquad (11)$$

and zero elsewhere. The scale-separating operator defined as $S^{pm} := P^p \circ R$ has the property of a projector indicated by the additional superscript p. The second prolongation operator considered in this paper yields a linear prolongation subject to

$$\bar{\mathbf{u}}_i^h = P^s \left[\bar{\mathbf{u}}_j^{\bar{h}}\right]_i = \bar{\mathbf{u}}_j^{\bar{h}} + \nabla^{\bar{h}} \bar{\mathbf{u}}_j^{\bar{h}} (\bar{\mathbf{r}}_j - \mathbf{r}_i) \qquad \forall \Omega_i \subset \bar{\Omega}_j \qquad (12)$$

and zero elsewhere. $\mathbf{r}_i$ and $\mathbf{r}_j$ denote the geometrical vectors pointing to the center of the child control volume Ω_i and the parent control volume $\bar{\Omega}_j$, respectively. $\nabla^{\bar{h}}$ designates the discrete gradient operator on the parent grid. Due to this, values from neighbouring parent control volumes and, consequently, child control volumes belonging to these neighbouring parent control volumes influence the final large-scale value in the child control volume Ω_i. Hence, P^s does not provide us with a projective scale-separating operation as shown in Gravemeier [6]. It rather produces a form of smoothing prolongation, which is indicated by the additional superscript s. The complete scale-separating operator is defined as $S^{sm} := P^s \circ R$. Nevertheless, S^{sm} exhibits a fundamentally different character compared to, e.g., discrete smooth filters. Alternative definitions for the restriction as well as the prolongation operator are certainly conceivable.

The validity of (8) with respect to the subgrid viscosity term in a complete sense remains to be analyzed. In Gravemeier [6], it was shown that discrete smooth filters in contrast to the scale-separating operators based on combined multigrid operators do not satisfy (8) in a strict sense due to the fact that the third term in line 3 of (8) cannot be represented. However, there is also a crucial difference between S^{pm} and S^{sm} in this context: there is no large-scale (subgrid) viscous flux across the small-scale boundary Γ' for S^{pm}. As a result, (8) may be specified for S^{pm} as

$$B_{NS}\left(\mathbf{v}^h, q^h;\ \mathbf{u}^h, p^h\right) - \left(\mathbf{v}'^h, \nu^T \mathbf{n} \cdot \nabla \mathbf{u}^h\right)_{\Gamma'} = \left(\mathbf{v}^h,\ \mathbf{f}\right)_\Omega \qquad (13)$$

In Fig. 1, the definition of large- and small-scale boundaries in a finite volume method is visualized for a 2-D case. The large-scale weighting function $\bar{\mathbf{v}}^{\mathrm{h}}$ is exclusively defined on the large-scale boundaries belonging to the parent control volume as shown in Fig. 1(a). The small-scale weighting function $\mathbf{v}'^h$ is exclusively defined on the inner boundaries of the child control volumes, confer Fig. 1(b).

4. Subgrid-scale modeling within the multiscale environment

It is focussed on the specific modification of the Smagorinsky [8] model restricting the dependence on the small scales subject to

$$\nu^T = (C_S h)^2 \left|\varepsilon\left(\mathbf{u}'^h\right)\right| = (C_S h)^2 \left|\varepsilon\left(\mathbf{u}^h - \bar{\mathbf{u}}^h\right)\right| \qquad (14)$$

which has been named 'small-small' model in Hughes et al. [1] and appears to be the most natural version within the multiscale environment.

The dynamic modeling procedure proposed in Germano et al. [9] enables a computation of the constant C_S as a function of time and position. It is interesting to note that the dynamic modeling procedure already distinguishes large resolved scales, small resolved scales, and unresolved scales explicitly. This amounts to be exactly the type of scale separation in the variational three-scale formulation. The dynamic modeling procedure based on the aforementioned scale-separating operators including alternative formulations is elaborated in Gravemeier [6].

5. Conclusions

A general class of scale-separating operators based on combined multigrid operators and suited for variational multiscale LES both with dynamic and constant-coefficient based subgrid-scale modeling has been proposed. Only one representative of the class exhibits the important property of a projector allowing to fulfill the theoretical assumption which underlies the scale separation within the variational multiscale method in a strict sense. The operators are further analyzed, implemented in a second-order accurate energy-conserving finite volume method, and tested for the case of a turbulent channel flow in Gravemeier [6].

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